1. Let f be a continuous function which takes real numbers as input and gives real numbers as output. (In mathematical shorthand, we write this as " $f : \mathbb{R} \to \mathbb{R}$ ".) A *fixed point* of f is an input x such that f(x) = x. Restrict the domain and range of f to some closed interval [a, b]: that is, our input x is always some number between a and b, and the output f(x) is always some number between a and b, and the output f(x) is always some number between a and b. Show that f has at least one fixed point.

(Hint: let g(x) = f(x) - x, plug in some points, and use an important theorem from class.)

- 2. The extreme value theorem tells us that, for a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on a closed interval [a, b], f achieves a maximum and a minimum on its range.
 - (a) Show that, on the *open* interval (a, b), f is not guaranteed to achieve a minimum or maximum.

(Hint: what's the difference between the intervals (a, b) and [a, b]?)

(b) Provide an example of a function that *never* achieves a maximum or minimum on its domain.

(Hint: think about limits at positive and negative infinity!)

- 3. Suppose that f'(x) = g'(x). Prove that f(x) = g(x) + C, where C is a real number.
- 4. (Bonus!) Given a differentiable function f : R → R, we can use its linear approximation L(x) at some point x₀ to guess the value of the nearest root x* of f. This guess is the point x₁ such that L(x₁) = 0. Setting L(x₁) = 0, we get

$$0 = f(x_0) + f'(x_0)(x_1 - x_0).$$

If we solve for x_1 , we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

(Try plugging x_1 into L(x) — you should get zero!) We can get an even *better* approximation of x^* by repeating these steps: the linear approximation of f at the point x_1 is given by

$$L(x) = f(x_1) + f'(x_1)(x - x_1),$$

and we can find the point x_2 where $L(x_2) = 0$ using the same process as above. This recursive algorithm of setting

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})},$$

where x_n is the n^{th} approximation of x^* , is called *Newton's method*.

- (a) Draw a picture to explain how Newton's method works.
- (b) Let $f(x) = x^3 x^2 2$. For $x_0 = 2$, find x_1 , x_2 , and x_3 . Using a tool like Desmos or Symbolab, find the exact value of the closest root to $x_0 = 2$. How far off were your three guesses? Does our initial guess x_0 affect how quickly our guesses get better?

Some history.

- 1. The first problem is a case of L.E.J. Brouwer's fixed point theorem, a seminal result linking analysis (and, by extension, calculus), topology, and abstract algebra during the formative years of modern analysis at the University of Göttingen. The University was the epicenter of mathematical development from the late 1800s until 1933, when the Nazis forcibly expelled scores of preeminent scholars because their theoretical work (disparagingly called "Jewish physics," based on the work of Albert Einstein) was antithetical to the Nazi party's *Deutsche Physik* and *Deutsche Mathematik* nationalist movements in the physical and mathematical sciences. According to mathematician Abraham Fraenkel, Brouwer positioned himself as the "champion of Aryan Germanness" and was outspoken against Eastern European Jewish (*Ostjuden*) authorship in Mathematische Annalen, the foremost German mathematics journal of the time.
- 2. The second result was originally proved by Bernard Bolzano, a leftist Catholic priest, during his exile. Bolzano also gave us the so-called "epsilon-delta" (symbolically, the " ε - δ ") definition of the limit, which underpins much of modern real analysis. Moreover, a version of the proof of the intermediate value theorem in your textbook was originally given by Bolzano in 1817, about 5,000 years after it was originally posited by Bryson of Heraclea.