Week 9 Recitation Problems

MATH:114, Recitations 309 and 310

Names: ____

An **improper integral** is the definite integral of a function where **one or both of the limits of integration approach infinity** or **the function is discontinuous somewhere on the interval of integration.**

1. Using what you've covered in lecture, fill out the table below. How do you know whether the integral is finite or infinite?

$p =$	Integral	Finite or Infinite?
1	$\int_1^\infty \frac{1}{x}\,dx$	infinite
$1/2$	$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$	infinite
2	$\int_1^\infty \frac{1}{x^2}\,dx$	finite

go to 0 "too slowly"

goes to 0 fast!

How does this work in terms of finite sums? how much does each chunk of finite terms add on?

2. If $p \neq 1$, compute the improper integral

$$\int_1^\infty \frac{1}{x^p}\,dx.$$

What are the conditions on p that determine the convergence of the integral? Why do they make sense?

$$\int_0^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{x^p}\,dt = \lim_{t\to\infty} \int_0^t x^{-p}\,dt = \lim_{t\to\infty} \left(\frac{1}{1-p} \cdot \frac{1}{t^{p-1}}\right) \Rightarrow \begin{cases} p>1, & \frac{1}{t^{p-1}} \longrightarrow 0 \\ p<1, & \frac{1}{t^{p-1}} \longrightarrow \infty \end{cases}$$

denominator gets big

denominator gets small

. . . .

Let's find out if $\int_3^\infty \ln(x)/\sqrt{x}\,dx$ is convergent.

3. Draw and compare the graphs of $f(x) = \ln(x)$ and $g(x) = 1$. When $x \geq 3$, which of the functions is greater than the other?

$\ln(x) > 1$ for all $x \geq 3$ (bonus: where are they equal?)

4. Using the result from Problem 4, what can you say about the functions $\ln(x)/\sqrt{x}$ and $1/\sqrt{x}$ when $x \geq 3$? If we integrate them as $\int_3^\infty \ln(x)/\sqrt{x}\,dx$ and $\int_3^\infty 1/\sqrt{x}\,dx$, which integral should be bigger?

$\ln(x) > 1$ for all $x \geq 3$ implies $\frac{\ln(x)}{\sqrt{x}} > \frac{1}{\sqrt{x}}$ for all $x \geq 3$

5. Compute the integral $\int_3^\infty 1/\sqrt{x}\,dx$. Based on your result, what can you say about $\int_3^\infty \ln(x)/\sqrt{x}\,dx$?

$\int_3^\infty \frac{1}{\sqrt{x}}dx = \lim_{t\to\infty} \int_3^t \frac{1}{\sqrt{x}}dx = \lim_{t\to\infty} \left(2\sqrt{x}\Big|_3^t \right) = \lim_{t\to\infty} 2\sqrt{t} - \lim_{t\to\infty} 2\sqrt{3} = \infty - 2\sqrt{3} = \infty \implies$ diverges.

because $\frac{\ln(x)}{\sqrt{x}} > \frac{1}{\sqrt{x}}$ for all $x \geq 3$, $\frac{\ln(x)}{\sqrt{x}}$ must diverge as well!

6. Suppose we have two functions $f(x)$ and $g(x)$, and let $f(x) \geq g(x) \geq 0$ where $x \geq a$. If...

$\int_a^\infty f(x)\,dx$ diverges $\implies \int_a^\infty g(x)\,dx$?

$\int_a^\infty f(x)\,dx$ converges $\implies \int_a^\infty g(x)\,dx$ converges

$\int_a^\infty g(x)\,dx$ diverges $\implies \int_a^\infty f(x)\,dx$ diverges

$\int_a^\infty g(x)\,dx$ converges $\implies \int_a^\infty f(x)\,dx$?

Note: the symbolic phrase $a \implies b$ means "if a, then b."

7. Think about the integral

$$\int_2^\infty \frac{\cos^2(t)}{t^2}\,dt.$$

Do you think that this integral converges or diverges? Why?

converges because $\cos^2(t)$ has min 0 and max 1, but $\int \frac{1}{t^2}dt$ converges on $[2, \infty)$.

8. What is a good function to compare the above integrand to? Write an inequality to justify your answer.

because $-1 \leq \cos(t) \leq 1$, $0 \leq \cos^2(t) \leq 1$. so, we have

$$\frac{\cos^2(t)}{t^2} \leq \frac{1}{t^2} \implies \int_2^\infty \frac{\cos^2(t)}{t^2}dt \leq \int_2^\infty \frac{1}{t^2}dt,$$

but because $\int_2^\infty \frac{1}{t^2}dt$ converges, so does $\int_2^\infty \frac{\cos^2(t)}{t^2}dt$.