Week 8 Recitation Problems

MATH:114, Recitations 309 and 310

Last week, we covered **linear** and **quadratic approximations** for a given function f. These approximations lead to **Taylor's Theorem**, which says:

Theorem (Taylor)**.** *Let f be continuously differentiable $N+1$ times at the point a. Then, there is a function $R(x)$ and a point c between a and x which satisfies the following equation:*

$$f(x) = P_N(x) + R_N(x),$$

where

$$P_N(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}_{\textit{quadratic approximation!}} + \cdots + \frac{f^N(a)}{N!}(x-a)^N,$$

and

$$R(x) = \frac{f^{N+1}(c)}{(N+1)!}(x-a)^{N+1}.$$

The function $P_N(x)$ is the N^{th}-order **Taylor polynomial** — that is, a polynomial of degree N which approximates f at a. $R_N(x)$ is the **remainder** or **error** function, and represents how far away $P_N(x)$ is from $f(x)$.

1. Let $f(x) = \sin(x)$ and $a = 0$. Compute the 6^{th}-order Taylor polynomial $P_6(x)$ and the remainder function $R_6(x)$. What pattern do you see?

x) = sin(x)
x) = cos(x)
x) = -sin(x)
) = -cos(x)
) = sin(x)

$$P_6(x) = \sin(0) + \cos(0)\cdot x - \frac{\sin(0)}{2!}\cdot x^2 - \frac{\cos(0)}{3!}\cdot x^3 + \frac{\sin(0)}{4!}x^4 + \frac{\cos(0)}{5!}x^5 - \frac{\sin(0)}{6!}x^6$$
$$= 0 + x - 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - 0$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

no sin terms – alternating odd powers!

$$R_6(x) = \frac{-\cos(c)}{7!}x^7$$

2. Suppose $f(x) = \sin(x)$ and $a = 0$, and that N is an arbitrary finite number. Write an expression for the N^{th}-order Taylor polynomial $P_N(x)$ of $f(x)$ using summation notation — that is,

$$P_N(x) = \sum_{k=1}^{N} \frac{f^k(x)}{k!}x^k$$

Also find the remainder function $R_N(x)$. Use the pattern you found in Problem 1 to help. **Once you finish this question, STOP and tell one of the instructors. We'll discuss our results as a class before moving on!**

$$P_N(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + R_{2k+1}(x),$$

$$f(x) = \sum_{k=1}^{N} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \frac{(-1)^{k+1}\,\mathrm{trig}(c)}{(2k+2)!}x^{2k+2}$$

with trig(c) is one of sin(c), cos(c), -sin(c), -cos(c).

3. Fill out the following table. **In the first column**, write one of the four expressions for $R_N(x)$ that we discussed as a class. **In the second column**, write the maximum possible value of the $f^{N+1}(c)$ that appears in $R_N(x)$. **In the third column**, write the absolute value of the value in the previous column. **In the fourth column**, substitute the absolute value in the previous question for $f^{N+1}(c)$ in the expression of $R_N(x)$. The first row of the table is filled out for you.

$R_N(x)$	$\max f^{N+1}(c)$	$\lvert \max f^{N+1}(c) \rvert$	$R_N(x)$ Upper Bound
$R_N(x) = \frac{\sin(c)}{(2k+2)!} x^{2k+2}$	$-1, 1$	1	$R_N(x) \leq \frac{(1)}{(2k+2)!} x^{2k+2} = \frac{x^{2k+2}}{(2k+2)!}$
$= \frac{\cos(c)}{(2k+2)!} x^{2k+2}$			
$= \frac{-\sin(c)}{(2k+2)!} x^{2k+2}$			
$= \frac{-\cos(c)}{(2k+2)!} x^{2k+2}$			

What do you notice about the upper bounds of $R_N(x)$?

4. Using a a visualizer like Desmos, graph the function

$$\frac{\left(\frac{\pi}{2}\right)^{x+1}}{(x+1)!}$$

Using this graph, what can you say about the function

$$\frac{x^{N+1}}{(N+1)!}$$

for any x, the numerator grows much slower than the denominator, so this goes to 0 at infinity!

as N goes to infinity? **Once you finish this question, STOP and tell one of the instructors. We'll discuss our results as a class!**

5. Based on our discussion, what can we say about the remainder $R_N(x)$ as N goes to infinity? Use this to conclude that

$$\sin(x) = \lim_{N\to\infty}\left[\sum_{k=1}^{N} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x)\right]$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \lim_{N\to\infty} R_{2k+1}(x)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + 0$$