Week 8 Recitation Problems

MATH:114, Recitations 309 and 310

Last week, we covered **linear** and **quadratic approximations** for a given function f. These approximations lead to **Taylor's Theorem**, which says:

Theorem (Taylor). *Let f be continuously differentiable $N+1$ times at the point a. Then, there is a function $R(x)$ and a point c between a and x which satisfies the following equation:*

$$f(x) = P_N(x) + R_N(x),$$

where

$$P_N(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}_{\text{quadratic approximation!}} + \cdots + \frac{f^N(a)}{N!}(x-a)^N,$$

and

$$R(x) = \frac{f^{N+1}(c)}{(N+1)!}(x-a)^{N+1}.$$

The function $P_N(x)$ is the N^{th}-order **Taylor polynomial** — that is, a polynomial of degree N which approximates f at a. $R_N(x)$ is the **remainder** or **error** function, and represents how far away $P_N(x)$ is from $f(x)$.

1. Let $f(x) = \sin(x)$ and $a = 0$. Compute the 6^{th}-order Taylor polynomial $P_6(x)$ and the remainder function $R_6(x)$. What pattern do you see?

2. Suppose $f(x) = \sin(x)$ and $a = 0$, and that N is an arbitrary finite number. Write an expression for the N^{th}-order Taylor polynomial $P_N(x)$ of $f(x)$ using summation notation — that is,

$$P_N(x) = \sum_{k=1}^{N} \frac{f^k(x)}{k!}x^k$$

Also find the remainder function $R_N(x)$. Use the pattern you found in Problem 1 to help. **Once you finish this question, STOP and tell one of the instructors. We'll discuss our results as a class before moving on!**

3. Fill out the following table. **In the first column**, write one of the four expressions for $R_N(x)$ that we discussed as a class. **In the second column**, write the maximum possible value of the $f^{N+1}(c)$ that appears in $R_N(x)$. **In the third column**, write the absolute value of the value in the previous column. **In the fourth column**, substitute the absolute value in the previous question for $f^{N+1}(c)$ in the expression of $R_N(x)$. The first row of the table is filled out for you.

$R_N(x)$	$\max f^{N+1}(c)$	$\lvert \max f^{N+1}(c) \rvert$	$R_N(x)$ Upper Bound
$R_N(x) = \frac{\sin(c)}{(N+1)!} x^{N+1}$	$-1, 1$	1	$R_N(x) \leq \frac{(1)}{(N+1)!} x^{N+1} = \frac{x^{N+1}}{(N+1)!}$

What do you notice about the upper bounds of $R_N(x)$?

4. Using a a visualizer like Desmos, graph the function

$$\frac{\left(\frac{\pi}{2}\right)^{x+1}}{(x+1)!}$$

Using this graph, what can you say about the function

$$\frac{x^{N+1}}{(N+1)!}$$

as N goes to infinity? **Once you finish this question, STOP and tell one of the instructors. We'll discuss our results as a class!**

5. Based on our discussion, what can we say about the remainder $R_N(x)$ as N goes to infinity? Use this to conclude that

$$\sin(x) =$$