Week 5 Recitation Problems

MATH:114, Recitations 309 and 310

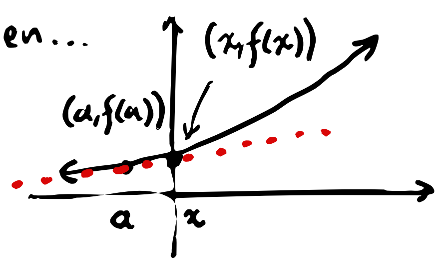
First, let's talk about **linear approximations.**

1. Let's approximate the function $f(x) = e^x$ around $x = 0$. Discuss a few ideas for this specific approximation.

how can we capture curvature? how far do we have to "zoom in" for our approximation to make sense? what makes an approximation good?

2. Let $L(x) = a_0 + a_1 x$. What kind of function does $L(x)$ describe? Try to approximate $f(x)$ at $x = 0$ by adjusting a_0 and a_1, and draw a picture of your approximation.

L(x) describes a line. pick a point a really close to x. then...



use a tangent line! we can say that...

$$f(x)=1,\ f'(x)=1 \Rightarrow L(x) = f(x) + \underbrace{f'(x)}_{\text{slope}}x$$

3. Come up with a general-purpose formula for approximating an arbitrary function $g(x)$ near the point x. *(Hint: what does the Mean Value Theorem say?)*

Yes! We can get, for a point a close to x,

$$L(x) = f(a) + \underbrace{f'(a)}_{\text{slope}}\underbrace{(x-a)}_{\text{distance from } x \text{ to } a}$$

uses the "point-slope" formula

Now, we can talk about **quadratic** and **higher-order** approximations. To do so, we're going to find a quadratic function $Q(x) = a_0 + a_1 x + a_2 x^2$ that approximates $f(x)$ near the point x. From the previous page, let

$$f(x) = e^x.$$

4. To construct $Q(x)$, we want the first *and* second derivatives to look a lot like the first and second derivatives of $f(x)$. In each of the following equations, set $f(x)$ (or its derivatives) equal to $Q(x)$ (or its derivatives) and solve for each coefficient.

First, match the values of the functions:

$$f(0) = Q(0) \implies a_0 = \underline{1}$$

$f(0) = a_0 + a_1(0) + a_2(0)^2$

$1 = a_0$

Then, match the values of the first derivatives:

$$f'(0) = Q'(0) \implies a_1 = \underline{1}$$

$f'(0) = a_1 + 2a_2(0)$

$1 = a_1$

Finally, match the values of the second derivatives:

$$f''(0) = Q''(0) \implies a_2 = \underline{\tfrac{1}{2}}$$

$f''(0) = 2a_2$

$1 = 2a_2$

$\frac{1}{2} = a_2$

5. Using the coefficients you just found, write out $Q(x)$. Can you come up with a general-purpose formula for the quadratic approximation $Q(x)$ for an arbitrary function $g(x)$? What about a cubic approximation?

$Q(x) = 1 + x + \frac{1}{2}x^2 \ldots$ using this pattern, we get

$$Q(x) = f(x) + f'(x)(x) + \frac{f''(x)}{2}(x)$$

or, at the point a,

$$Q(a) = f(a) + f'(x)(x-a) + \frac{f''(x)}{2}(x-a)$$

6. Find the cubic approximation $C(x)$ for $f(x) = e^x$. For each of the linear, quadratic, and cubic approximations, check its value against the true value of $f(x)$ at $x = 0$.

$$C(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$