Week 5 Recitation Problems

MATH:114, Recitations 309 and 310

Logarithms and Exponential Change

1. Evaluate

$$\int \frac{1}{x\ln(x)}\,dx$$

$$\int \frac{1}{x\ln(x)}\,dx$$

$$= \int \frac{1}{x}\cdot\frac{1}{\ln(x)}\,dx$$

$$\left\{ \begin{aligned} u &= \ln(x) \\ \frac{du}{dx} &= \frac{1}{x} \end{aligned} \right.$$

$$= \int \frac{du}{dx}\cdot\frac{1}{u}\,dx$$

$$= \int \frac{1}{u}\,du$$

$$= \ln(u) + C$$

$$= \ln(\ln(x)) + C$$

:)

2. Evaluate

$$\int \frac{x}{x^2+4}\,dx$$

use u-sub again!

$u = x^2+4$,

$\frac{du}{dx} = 2x$,

$du = 2x\,dx$

$$\int \frac{x}{x^2+4}\,dx$$

$$= \int \frac{\frac{1}{2}\frac{du}{dx}}{u}\,dx$$

$$= \frac{1}{2}\int \frac{1}{u}\,dx$$

$$= \frac{1}{2}\ln(u) + c$$

$$= \frac{1}{2}\ln(x^2+4) + C$$

:)

If a function $y(t)$ is increasing or decreasing at an exponential rate, we can say it is **exponentially growing** or **exponentially decreasing**, and this rate of change is proportional to its value at a time t. In other words, $y(t)$ is **proportional to its own derivative** $y'(t)$, so

$$\frac{d}{dt}\, y(t) = k \cdot y(t). \tag{$*$}$$

Writing this just in terms of our function y, and treating it like a variable, the following expressions are equivalent:

$$\frac{dy}{dt} = k \cdot y(t) \qquad y'(t) = k \cdot y(t) \qquad y' = k \cdot y$$

A **differential equation** is when a function is equated to its own derivative(s), in an expression like the ones above. An **initial value problem** arises when you are given k and the value of $y(t)$ at a "starting value" or **initial condition** t_0 on its domain — like $t_0 = 0$, so $y(t_0) = y(0) = C$, where C is some constant — and we are tasked with recovering the function $y(t)$. The above types of initial value problems have one specific solution:

$$y(t) = Ce^{kt},$$

where $k > 0$.

3. Check that the function $y(t) = Ce^{kt}$ satisfies the equation in $(*)$.

$\frac{d}{dt} y(t) = \frac{d}{dt} Ce^{kt}$

$= C \cdot k \cdot e^{kt}$,

so if $L = C \cdot k$, L is a constant, and

$\frac{dy}{dt} = L \cdot e^{kt}$ ✓

4. Find the general solution for the initial value problem where $k = 1/4$ and $y(t_0) = y(0) = 200$.

$\frac{dy}{dt} = \frac{1}{4} y$

Separation of variables:

$\frac{dy}{y} = \frac{1}{4} dt$

$\int \frac{1}{y} dy = \int \frac{1}{4} dt$

$\ln y = \frac{1}{4} t + C_0$

$e^{\ln y} = e^{\frac{1}{4} t + C_0}$

$y = e^{\frac{1}{4} t + C_0}$ ...

now, we find C

$200 = y(t_0)$

$= y(0)$

$= e^{\frac{1}{4}(0) + C_0}$

$= e^{C_0}$

let $e^{C_0} = C$,

so

$200 = e^{C_0} = C$,

$y(t) = 200 e^{\frac{1}{4} t}$

5. If $y(t)$ from Problem 4 describes the population of mosquitoes, when will we triumph over our pestilent insect overlords and vanquish their population? (In other words, at what time t does $y(t) = 0$?)

$0 = 200 e^{\frac{1}{4} t}$, but then either $200 = 0$ or $e^{\frac{1}{4} t} = 0$, neither of which are possible... the mosquitoes never die.