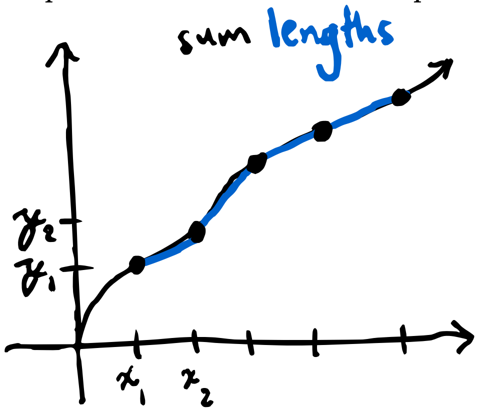
Week 5 Recitation Problems

MATH:114, Recitations 309 and 310

Curve Length and Surface Area

1. Given a function $f(x)$, how might we **approximate** the length of $f(x)$ on the closed interval $[a, b]$? Draw an annotated picture or write a few words to explain, and include relevant geometric formulas or ideas. *(Hint 1: use the Euclidean distance formula, which you are free to look up. Hint 2: break the curve up into chunks!)*



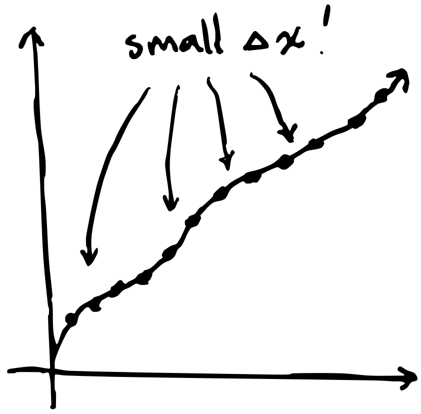
interpolate the curve, then sum the lengths of the segments using the distance formula

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\left(L = \sqrt{(\Delta x)^2 + (\Delta y)^2} \right)$$

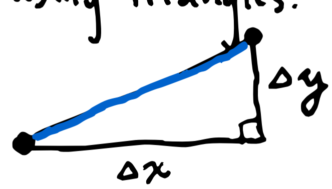
2. Using your strategy from Problem 1, translate your approximation into an exact continuous calculation (that is, one which uses an integral). Draw an annotated picture or write a few words to explain, and include relevant calculus theorems or geometric ideas. *(Hint: think about the rectangle or trapezoid methods for estimating the area under a curve, which you are free to look up.)*

shrink the distances between the x values! (or, make Δx smaller!)



m is our derivative! (slope of the line)

using triangles:



$\Delta y = m \Delta x$ (a line!)

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{(\Delta x)^2 + (m \Delta x)^2}$$

$$= \sqrt{1 + m^2}\, \Delta x$$

shrink Δx to dx to get

$$L = \sqrt{1 + f'(x)^2}\, dx.$$

using mvt:

$$f'(c) = \frac{y_2 - y_1}{x_2 - x_1} \Big\} \text{slope}$$

$$= \frac{\Delta y}{\Delta x},$$

$$f'(c)\Delta x = \Delta y$$

"there is a point c between x_1 and x_2 where the deriv is the slope" !!!

subbing,

$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{\Delta x^2 + (f'(c)\Delta x)^2}$$

$$= \sqrt{1 + f'(c)^2}\, \Delta x$$

3. Let

$$f(x) = \frac{x^3}{6} + \frac{1}{2x}.$$

Find the length of $f(x)$ when $1 \leq x \leq 3$.

$$\frac{d}{dx} f(x) = \frac{x^2}{2} - \frac{1}{2x^2} = \frac{x^4 - 1}{2x^2} = f'(x)$$

$$L = \int_1^3 \sqrt{1 + f'(x)^2}\, dx$$

$$= \int_1^3 \sqrt{\frac{4x^4}{4x^4} + \frac{x^8 - 2x^4 + 1}{4x^4}}\, dx$$

$$= \int_1^3 \sqrt{\frac{x^8 + 2x^4 + 1}{4x^4}}\, dx$$

$$= \int_1^3 \sqrt{\frac{(x^4+1)^2}{(2x^2)^2}}\, dx$$

$$= \int_1^3 \frac{x^4+1}{2x^2}\, dx$$

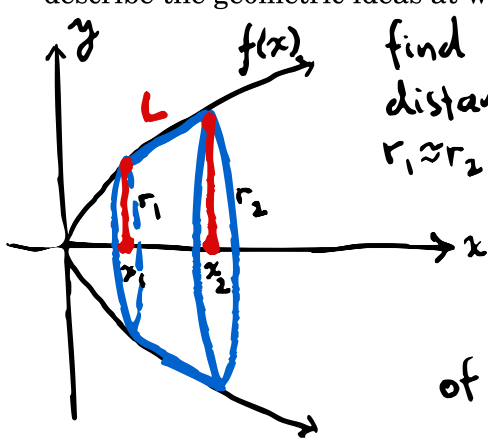
$$= \int_1^3 \frac{x^4}{2x^2} + \frac{1}{2x^2}\, dx$$

$$= \frac{x^3}{6} - \frac{1}{2x} \Big|_1^3 = \frac{14}{3}$$

4. The formula

$$S = 2\pi \int_a^b g(x) \cdot \sqrt{1 + g'(x)^2}\, dx$$

describes how to find the surface area of the solid generated by the curve $g(x)$ on the closed interval $[a, b]$. What is familiar about this formula? Using annotated pictures or a few words, describe the geometric ideas at work here.



find surface area of this shape: if we shrink the distance between x_1 and x_2 to be infinitesimal, then $r_1 \approx r_2$ and we find the surface area of a cylinder!

we know the length of L is given by

$L = \sqrt{1 + f'(x)^2}\, dx$, so we find the surface area of the cylinder by

$$2\pi \cdot r \cdot h = 2\pi \cdot g(x) \cdot L,$$

then add the areas up!

5. Let $g(x) = \sqrt{4 - x^2}$, and $-1 \leq x \leq 1$. Find the surface area of the solid generated by rotating $g(x)$ around the x axis.

$$\frac{d}{dx} g(x) = \frac{-x}{\sqrt{4 - x^2}},$$

$$S = \int_{-1}^{1} 2\pi \cdot g(x) \cdot \sqrt{1 + g'(x)^2}\, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \cdot \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \cdot \frac{2}{4 - x^2}\, dx$$

$$= 2\pi \int_{-1}^{1} 2\, dx$$

$$= 4\pi \Big|_{-1}^{1}$$

$$= 8\pi$$