Week 5 Recitation Problems

MATH:114, Recitations 309 and 310

Euler's Formula and Hyperbolic Functions

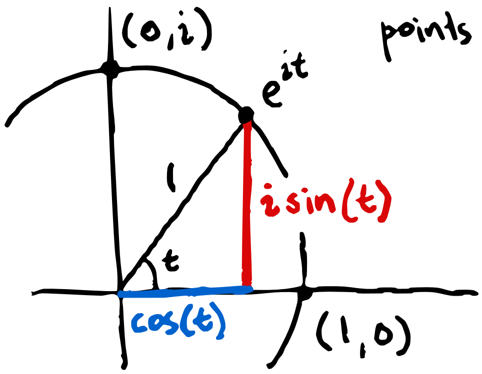
Euler's Formula, written as

$$e^{it} = \cos t + i \sin t,$$

where i is the complex unit $i = \sqrt{-1}$, is an expression which establishes the relationship between continuous complex growth (e^{it}) and trigonometry in the complex plane ($\cos t + i \sin t$).

1. Using an annotated picture or words, describe what Euler's formula tells us. *(Hint: think about the complex unit circle — that is, the circle centered at the origin which intersects the complex (vertical) axis at the points* $(0, i)$ *and* $(0, -i)$ *and the real (horizontal) axis at the points* $(1, 0)$ *and* $(-1, 0)$*. If* t *measures an angle counter-clockwise from the origin, where does each side of the equation end up?)*

Euler's formula gives us two equivalent ways to locate points on the unit circle.



$$\underbrace{\cos(t)}_{\text{real }(x)\text{ coord}} + \overbrace{i\sin(t)}^{\text{complex }(i)\text{ coord}}$$

$e^{it} = (e^{i})^{t}$

↳ e^{i} = continuous imaginary growth

↳ t = units of rotation

} traces out a circle!

$\Rightarrow e^{it}$ = continuous imaginary growth for t units of rotation!

2. Verify **Euler's Identity**

$$e^{i\pi} = -1$$

$e^{it} = \cos(t) + i\sin(t)$

$e^{i(\pi)} = \cos(\pi) + i\sin(\pi)$

$= -1 + i(0)$

$= -1,$

so $e^{i\pi} = -1$

recall that $i = \sqrt{-1}$, so $i^2 = -1$

3. (From a previous exam.) The derivative with respect to t of the complex function

$$e^t \cos(2t) + ie^t \sin(2t)$$

is

$$e^t \cos(2t) - 2e^t \sin(2t) + ie^t \sin(2t) + 2ie^t \cos(2t).$$

Show that this derivative is equal to the derivative with respect to t of $e^{(1+2i)t}$.

$$= e^t\left(\cos(2t) + i\sin(2t)\right) + 2e^t\left(i\cos(2t) - \sin(2t)\right)$$

$$= \text{"} \quad \text{"} \quad + 2e^t\left(i\cos(2t) + (i^2)\sin(2t)\right)$$

$$= \text{"} \quad \text{"} \quad + 2e^t i\left(\cos(2t) + i\sin(2t)\right)$$

$$= e^t\left(\left(\cos(2t) + i\sin(2t)\right) + 2i\left(\cos(2t) + i\sin(2t)\right)\right)$$

$$= e^t\left(e^{i2t} + 2ie^{i2t}\right)$$

$$= e^t\left((1+2i)e^{i2t}\right)$$

$$= (1+2i)e^{t+i2t}$$

$$= (1+2i)e^{(1+2i)t}$$

$$= \frac{d}{dt} e^{(1+2i)t} \quad \checkmark$$

4. (From a previous exam.) If we define

$$\sinh(t) = \frac{e^t - e^{-t}}{2} \qquad \text{and} \qquad \cosh t = \frac{e^t + e^{-t}}{2},$$

show that

$$(\cosh t)^2 - (\sinh t)^2 = 1$$

for all values of t.

$$\left(\frac{e^t - e^{-t}}{2}\right)^2 = \frac{e^{2t} + e^{-2t} - 2}{4}$$

$$\left(\frac{e^t + e^{-t}}{2}\right)^2 = \frac{e^{2t} + e^{-2t} + 2}{4}$$

$$\frac{e^{2t} + e^{-2t} + 2 - (e^{2t} + e^{-2t} - 2)}{4}$$

$$= \frac{4}{4}$$

$$= 1 \quad \checkmark$$

5. Using the definitions of cosh and sinh from Problem 4, show that

$$\cosh(2t) = (\cosh t)^2 + (\sinh t)^2$$

$$\frac{e^{2t} + e^{-2t} + 2 + (e^{2t} - e^{-2t} - 2)}{4}$$

$$= \frac{2e^{2t} + 2e^{-2t}}{4}$$

$$= \frac{e^{2t} + e^{-2t}}{2} = \cosh(2t)$$