Suppose that

$$f(x) = 2x^2$$

and

$$g(x) = x^3.$$

We are going to find the **volume of the solid generated by rotating these curves** using **two different methods.**

the disk and washer methods

The **disk method** is based on **finding the areas of infinitely thin donuts, or "washers,"** and adding those areas up.

To add those areas up, we use **an integral**. If we

(i) know our **bounds of integration**,
(ii) are **rotating our solid around the x axis**, and
(iii) **have two functions** f_{top} and f_{bottom},

our integral to find the volume is

$$V_{\text{washer}} = \int_a^b \pi(\underbrace{f_{\text{top}}(x)}_{\text{radius of outer circle}})^2 - \pi(\overbrace{f_{\text{bottom}}(x)}^{\text{radius of inner circle}})^2 dx,$$

which can be re-written as

$$V_{\text{washer}} = \pi \int_a^b (\underbrace{f_{\text{top}}(x)}_{\text{radius of outer circle}})^2 - (\overbrace{f_{\text{bottom}}(x)}^{\text{radius of inner circle}})^2 dx$$

What happens if we don't have two functions?

Let's apply it.

We set

$$f(x) = f_{\text{outer}}(x), \quad g(x) = f_{\text{inner}}(x)$$

(because f is "on top" of g) and find where the curves intersect — that is, where the **curves hit the same value.** This gives us our **bounds of integration.**

$$f(x) = g(x)$$

$$2x^2 = x^3$$

$$2 = \frac{x^3}{x^2}$$

$$2 = x$$

so the curves intersect at the points

$$(2, f(2)) = (2, g(2)) = (2, 8)$$

and

$$(0, f(0)) = (0, g(0)) = (0, 0)$$

We will integrate from $x = 0$ to $x = 2$.

Setting up our integral, we get

$$\begin{aligned}
V_{\text{washer}} &= \pi \int_0^2 f(x)^2 - g(x)^2 dx \\
&= \pi \int_0^2 (2x^2)^2 - (x^3)^2 dx \\
&= \pi \int_0^2 4x^4 - x^6 dx \\
&= \pi \left[\frac{4}{5}x^5 - \frac{1}{7}x^7\right]\Bigg|_0^2 \\
&= \frac{256\pi}{35}
\end{aligned}$$

the shell method

The **shell method** finds the **surface area of infinitely thin cylinders** and adds them up to find a volume. Recall that the surface area of a cylinder *without a top or bottom* (like a straw) is

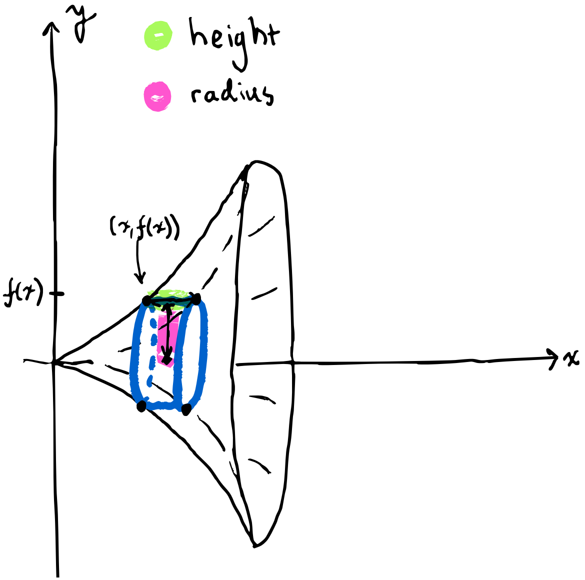
$$S = 2\pi \cdot r \cdot h$$

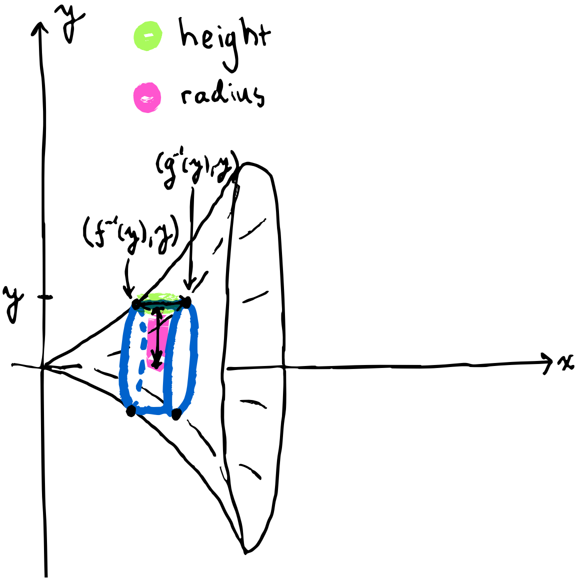
To add those areas up, we use **an integral** again.

If we

(i) know our **bounds of integration**,
(ii) are **rotating our solid around the x axis**, and
(iii) **have two functions f_{top}^{-1} and f_{bottom}^{-1}**,

we have good information. However, we need to **change our perspective** — if we are rotating around the x-axis, then **our cylinders are in terms of y.** (Why?)


y
height
radius
(x, f(x))
f(x)
x



y
height
radius
$(g^{-1}(y),y)$
$(f^{-1}(y),y)$
y
x

We then find these **inverse functions** by setting up equations and **solving for** x:

$$
\begin{aligned}
y &= 2x^2 \\
\frac{y}{2} &= x^2 \\
\sqrt{\frac{y}{2}} &= x = f^{-1}(y),
\end{aligned}
$$

$$
\begin{aligned}
y &= x^3 \\
\sqrt[3]{y} &= x = g^{-1}(y)
\end{aligned}
$$

Now, because the height and radius of our cylinders are in terms of y, we integrate with...

Our integral is then

$$V_{\text{shell}} = \int_a^b 2\pi \cdot \underbrace{y}_{\text{radius of cylinder}} \cdot \overbrace{(f_{\text{top}}^{-1}(y) - f_{\text{bottom}}^{-1}(y))}^{\text{height of cylinder}} \, dy,$$

which we can re-write as

$$V_{\text{shell}} = 2\pi \int_a^b y \cdot (f_{\text{top}}^{-1}(y) - f_{\text{bottom}}^{-1}(y)) \, dy$$

Let's set up our integral!

$$
\begin{aligned}
V_{\text{shell}} &= 2\pi \int_a^b y \cdot (f_{\text{top}}^{-1}(y) - f_{\text{bottom}}^{-1}(y))dy \\
&= 2\pi \int_0^8 y \cdot \left(\sqrt[3]{y} - \sqrt{\frac{y}{2}} \right) dy \\
&= 2\pi \left[-\frac{\sqrt{2}y^{\frac{5}{2}}}{5} - \frac{3y^{\frac{7}{2}}}{7} \right] \Bigg|_0^8 \\
&= \frac{256\pi}{35},
\end{aligned}
$$

so we get the **same result**!

questions?