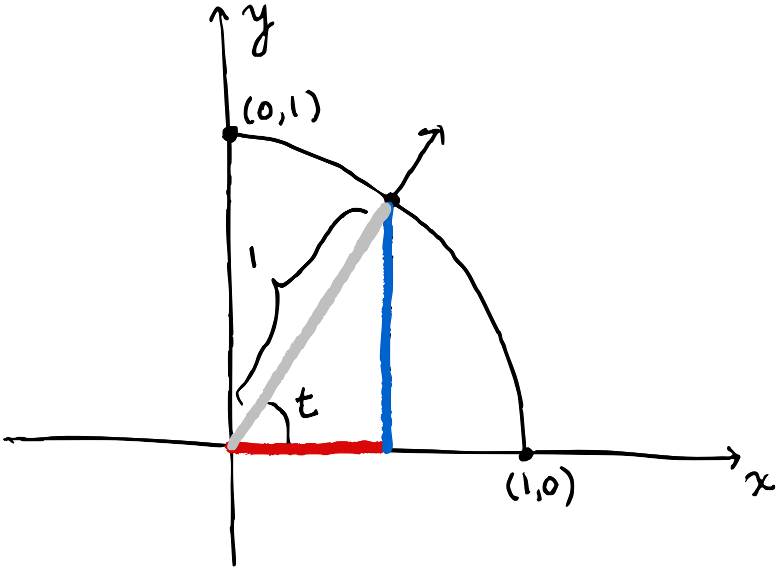
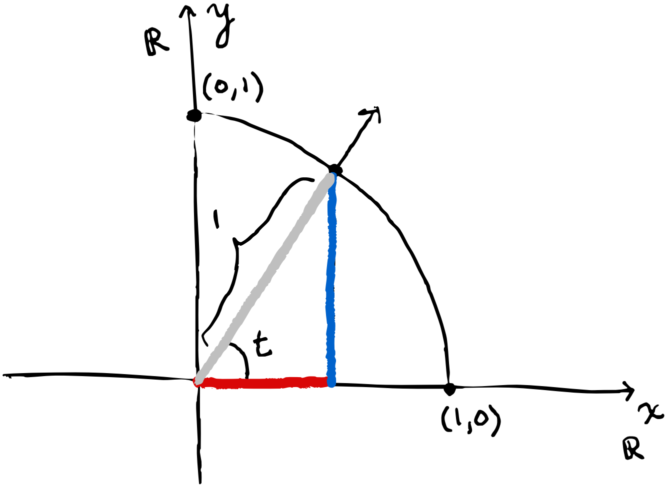
Euler’s identity and hyperbolic functions

Polar coordinates tell us where something is using an **angle from the origin** and a **stretching factor**.



y
(0,1)
1
t
(1,0)
x



● $1 \cdot \cos(t)$ ● $1 \cdot \sin(t)$

Because complex numbers are written as

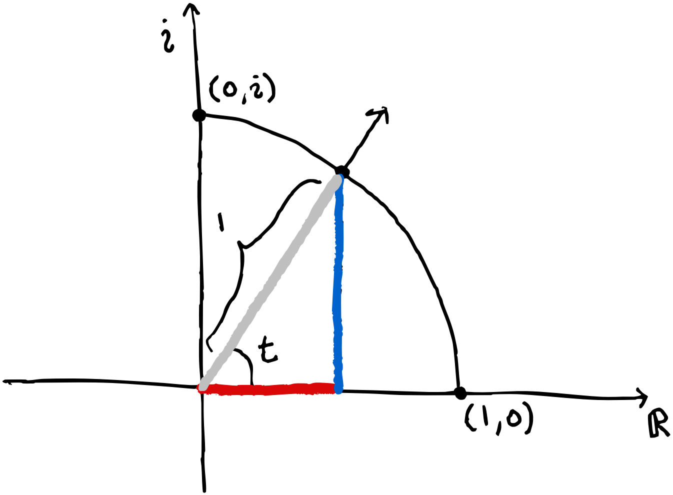
$$a + bi,$$

with a a real number and b a stretching (scaling) factor on the complex number i, we can say that

a represents **stretching in the x direction**

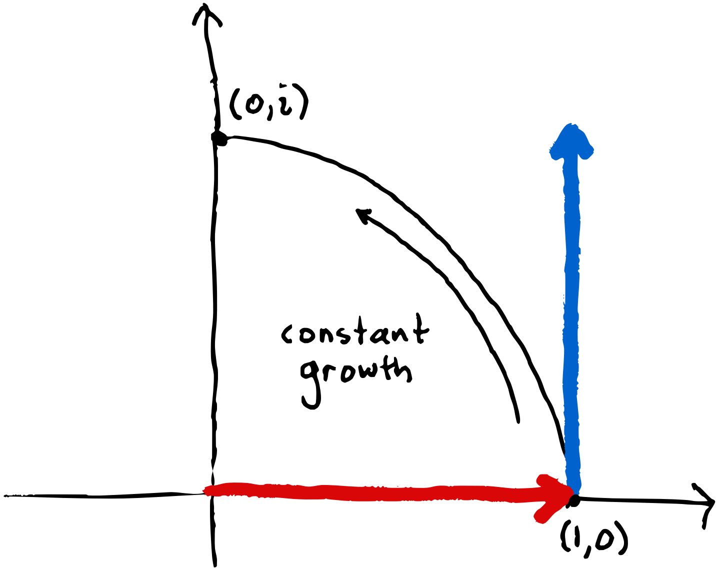
and

bi represents **stretching in the i direction**.



$1 \cdot \cos(t)$ $i \cdot \sin(t)$

... but how does e factor into this?



(0,i)
constant growth
(1,0)

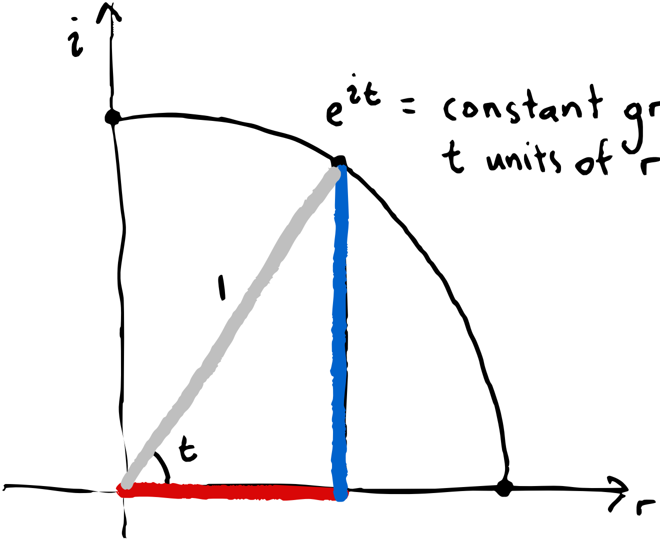
it's e^x

if we want to represent a point on the complex unit circle by **scaling** and **rotation**, we can write this as

$$e^{a+bi} = e^a \cdot e^{bi}.$$

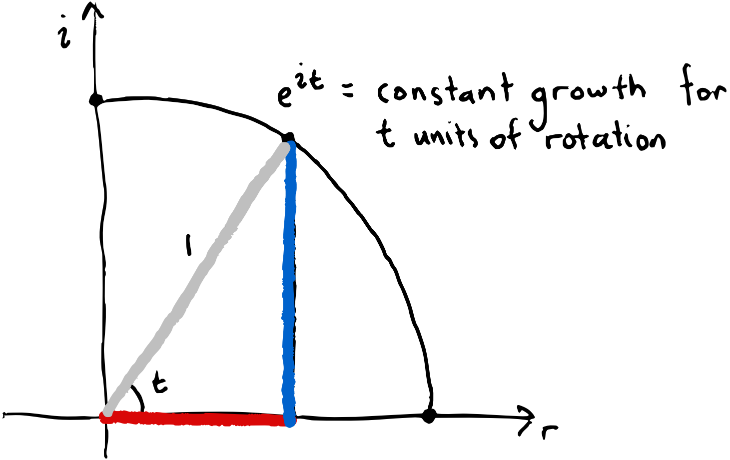
but if we translate our $a + bi$ into **polar coordinates** with radius 1, we get

$$e^{1 \cdot i \cdot t} = \underbrace{e^i}_{\text{rotation}} \cdot e^t$$



e^{it} = constant growth for t units of rotation

... and because we express polar coordinates in terms of $\sin$ and $\cos$...



$1 \cdot \cos(t)$ $\quad$ $i \cdot \sin(t)$

so our expressions

$$e^{ix}$$

and

$$\cos x + i \sin x$$

represent *the same thing!*

the *unit hyperbola*

$$x^2 - y^2 = 1$$

is like a circle because it **grows at precisely the same rate everywhere**, so we can define analogous trig functions on it!

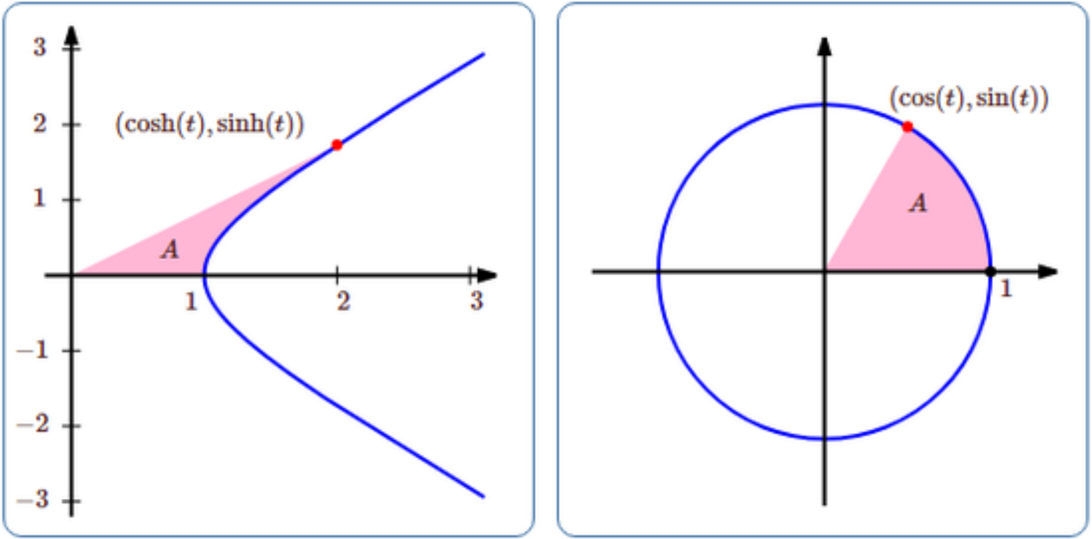


Figure *Geometric definitions of sin, cos, sinh, cosh: t is twice the shaded*