Week 12 Recitation Problems

MATH:114, Recitations 309 and 310

Names: Solutions

A *random variable* is a function that assigns numbers to outcomes of an experiment. Let's use coin tosses as an example: the possible outcomes, or the *sample space*, is the set $\Omega = \{H, T\}$, for Heads and Tails. We can set up C to be a random variable that models a coin-flipping game: if the coin turns up heads, I get two dollars, and otherwise I gain one dollar. Our random variable looks like this:

$$C = \begin{cases} 1 & \text{if the coin lands on } T \\ 2 & \text{if the coin lands on } H \end{cases}$$

We can also assign *probabilities* to each value of C: for example, the probability that our coin lands on H (or that we get two dollars) is

$$\mathbf{P}[C = 2] = \frac{1}{2}.$$

Because the probabilities of all values of C have to sum to 1, we also know that

$$\mathbf{P}[C = 1] = \frac{1}{2}.$$

The *expected value* of a random variable is a long-term average: each time I play the game, how much money can I expect to win? Because we have 2 possible values for our random variable, the expected value is

$$\sum_{i=1}^{2} i \cdot \mathbf{P}[C = i]$$

1. What is the expected value of the random variable C? In other words, how much should money should I expect to win every time I play the game?

$$\mathbb{E}[C] = \sum_{i=1}^{2} i \cdot \mathbb{P}[C=i] = 1 \cdot \mathbb{P}[C=1] + 2 \cdot \mathbb{P}[C=2]$$

$$= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{1}{2} + 1 \boxed{= \frac{3}{2}} \text{ or } \boxed{\$1.50}$$

2. If I play the game 1000 times, how much money should I expect to win?

Because the events (coin flips) are independent, the expectation of the sum is the sum of the expectations:

$$\mathbb{E}[X+X] = \mathbb{E}[X] + \mathbb{E}[X] \implies \mathbb{E}[1000 \cdot X] = 1000 \cdot \mathbb{E}[X] = \boxed{\frac{3}{2} \cdot 1000} = \boxed{\$1500}$$

Now, because I'm interested in winning some serious cash, I want to know something in particular: how many flips do I need before I get my first H? Let's define our random variable together.

3. What are the possible "strings" of coin flips that have all Ts and then one H? In other words, what is my *sample space*? I've started the list for you:

$$\Omega = \{H, TH, TTH, \underbrace{TTT}H, \underbrace{TTTT}H, \ldots$$

How many possible outcomes are there?

the number of Ts keeps increasing, so there are a $\boxed{\text{countably infinite}}$ number of possible outcomes!

for example, it's possible to flip 1000 tails before getting an H.

4. Let's make a random variable called **Flips**, and define it like this:

probability of flipping an $\boxed{H : 1/2}$

$\boxed{T : 1/2}$

$$\textbf{Flips} = \begin{cases} 1 & \text{one flip until the first head} \\ \vdots & \vdots \\ n & n \text{ flips until the first head} \\ \vdots & \vdots \end{cases}$$

What is the probability of getting TTH — that is, **Flips** $= 3$? What is the probability that **Flips** $= n$, for some natural number n?

each coin flip is independent, so

$$P[T.T.H] = P[T.] \cdot P[T] \cdot P[H] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \boxed{= \frac{1}{8}}$$

5. Let's check that our probabilities sum to 1. The *partial sum* S_n is the sum of our ~~first n~~ terms in our sequence of probabilities. For example,

this is a $\boxed{\text{geometric series.}}$

$$S_3 = \mathbf{P}[\textbf{Flips} = 1] + \mathbf{P}[\textbf{Flips} = 2] + \mathbf{P}[\textbf{Flips} = 3]$$
$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

rewrite as $S_3 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$

Can you express S_3 where only *one* term has an exponent? *(Hint: multiply S_3 by $1/2$, then subtract the result from S_3, and do some algebra.)* Can you express S_n the same way?

$$S_3 - \frac{1}{2}S_3 = \left(\frac{1}{2} + \cancel{\frac{1}{2^2}} + \cancel{\frac{1}{2^3}}\right) - \left(\cancel{\frac{1}{2^2}} + \cancel{\frac{1}{2^3}} + \frac{1}{2^4}\right) = \frac{1}{2} - \frac{1}{2^4}$$

$$\Rightarrow S_3 - \frac{1}{2}S_3 = \frac{1}{2} - \frac{1}{2^4}$$
$$S_3\left(1 - \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2^4}$$
$$\boxed{S_3 = \frac{\frac{1}{2} - \frac{1}{2^4}}{1 - \frac{1}{2}}} \quad \boxed{S_n = \frac{\frac{1}{2} - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}}}$$

6. Take the limit of your expression for S_n. Using that limit, what can we say about the sum of the probabilities as n goes to infinity?

$$\lim_{n\to\infty} \frac{\frac{1}{2} - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = \frac{\frac{1}{2} - 0}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} \boxed{= 1} \quad \text{OR} \quad \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1 - 0 \boxed{= 1}$$

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7. Write out the first few terms of $\mathbf{E}[\textbf{Flips}]$, the expected value of **Flips**. If $p_i = \mathbf{P}[\textbf{Flips} = i]$, we can write it like:

$$\mathbf{E}[\textbf{Flips}] = (1 \cdot p_1) + (2 \cdot p_2) + (3 \cdot p_3) \cdots$$
$$= (p_1) + (p_2 + p_2) + (p_3 + p_3 + p_3) + \cdots$$
$$= (p_1 + p_2 + p_3 + \cdots) + (p_2 + p_3 + \cdots) + (p_3 + \cdots) + \cdots$$

grouped into partial sums with the first term missing!

where each of the terms are grouped into partial sums. We know the value of $p_1 + p_2 + p_3 + \cdots$ (look at the last problem)! What is the value of $p_2 + p_3 + \cdots$?

$$p_1 + p_2 + p_3 + \cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$
$$= \lim_{n\to\infty} S_n \boxed{= 1}$$

so, $p_2 + p_3 + p_4 + \cdots = \underbrace{(p_1 + p_2 + p_3 + \cdots)}_{=1} - \underbrace{p_1}_{=\frac{1}{2}}$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

8. What is the expected value of **Flips**?

$$E[X] = \underbrace{(p_1 + p_2 + p_3 + \cdots)}_{=1} + \underbrace{(p_2 + p_3 + p_4 + \cdots)}_{=\frac{1}{2}} + \underbrace{(p_3 + p_4 + p_5 + \cdots)}_{=\frac{1}{4}}$$

$$= 1 + \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots}_{=1}$$

$= 1 + 1 \boxed{= 2}$, so we should only wait, in expectation, $\boxed{\text{2 flips to see an H.}}$