sequences let us do calculus!

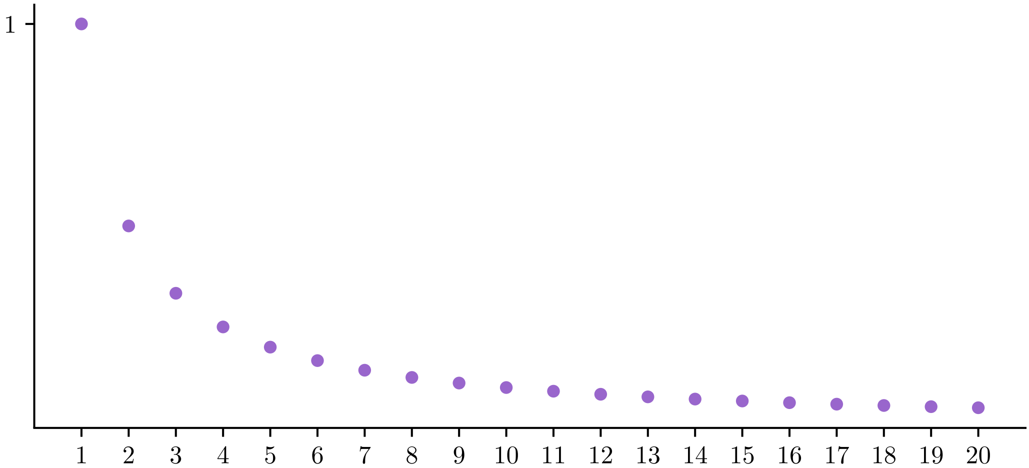
a *sequence* is an *indexed set* of *objects*.

$$\{a_n\}_{n=1}^{N} = \{a_1, a_2, a_3, \ldots, a_N\}$$

we can make sequences *in any space we want*,
not just the real numbers $\mathbb{R}$.

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$$

$$\left\{\frac{1}{n}\right\}_1^\infty = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$$


1
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17
18
19
20

a sequence *has a limit at* L if the entries a_n get *arbitrarily close to* L as $n \to \infty$.

$$\lim_{n\to\infty} a_n = L$$

when:

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$\hookrightarrow$ for every positive number ϵ...

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$\hookrightarrow$ for every index m bigger than N...

$\hookrightarrow$ the distance from a_m to L is less than ϵ.

mathematically...

$$\lim_{n\to\infty} a_n = L$$

$$\iff$$

$$\forall\epsilon > 0\,,\ \exists N \in \mathbb{N}\ \text{ s.t. }\ \forall m > N\,,\ |a_m - L| < \epsilon$$

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$$\frac{1}{m} = a_m \qquad a_N = \frac{1}{10}$$

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if we choose $N = 10$, then for $m > 10$,

$$\frac{1}{m} = a_m < a_N = \frac{1}{10}$$

$$
\begin{aligned}
|a_N - L| &= \left| \frac{1}{10} - 0 \right| \\
&= \frac{1}{10} \\
&= \epsilon
\end{aligned}
$$

but...

$$|a_m - L| = \left|\frac{1}{m} - 0\right|$$

but...

$$
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&< \left|\frac{1}{10} - 0\right|
\end{aligned}
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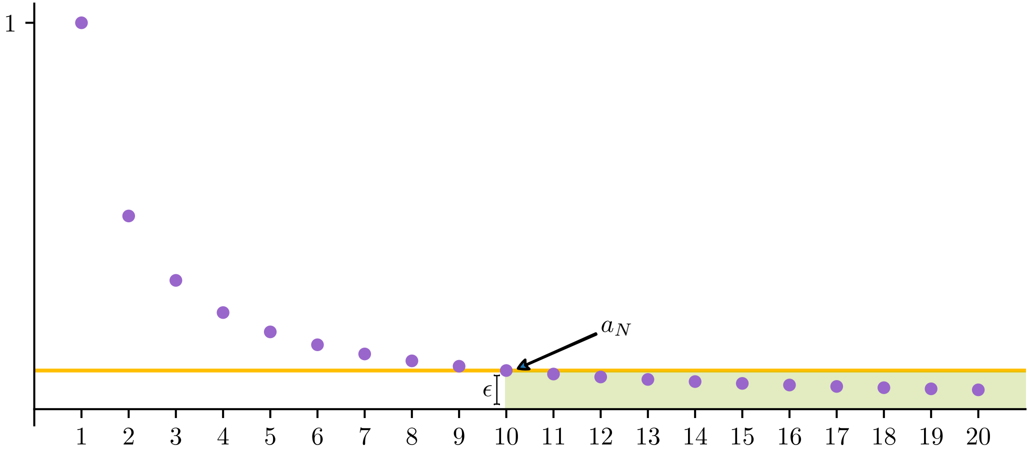
$$|a_m - 0| < \epsilon$$


1
a_N
ϵ
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

if we can do this for *every* $\epsilon > 0$...

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then our sequence converges to 0.

big picture:

our sequence has a limit at L if the elements *eventually get really close to* L.

an important type of sequence is called a *Cauchy* sequence.

in a Cauchy sequence, the elements get arbitrarily close to *each other.*

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mathematically...

$$\{a_n\}_{n=1}^{\infty} \text{ is Cauchy}$$

$$\iff$$

$$\forall \epsilon > 0\,,\ \exists N \in \mathbb{N} \text{ s.t. } \forall p, q > N\ ,\ |a_p - a_q| < \epsilon$$

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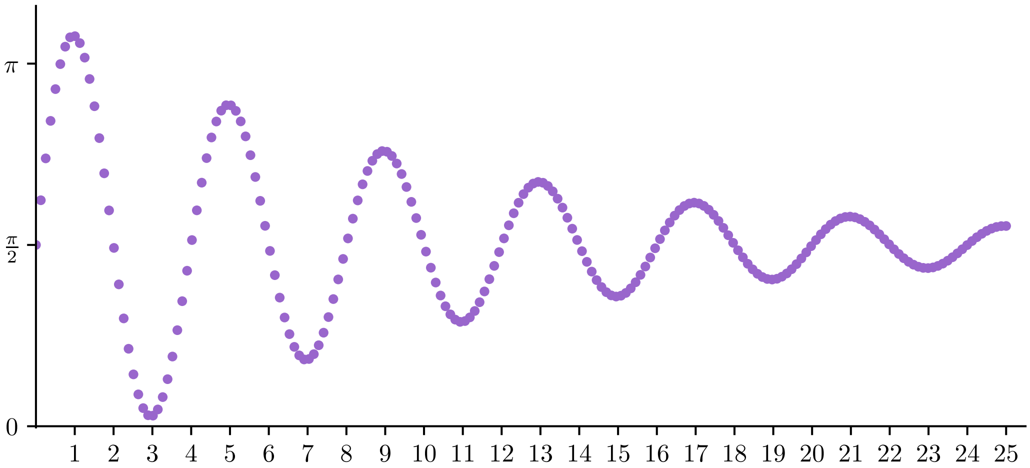
$$\left\{\sin\left(\frac{\pi \cdot n}{2}\right) \cdot e^{\frac{1/10}{n}}\right\}_{n=0}^{\infty}$$



π
$\frac{\pi}{2}$
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

$$N = 2, \epsilon = \frac{\pi}{2}$$

$$\implies |a_p - a_q| < \frac{\pi}{2}$$

(when p and q are bigger than 2)

let $p = 5$ and $q = 7$:

$$|a_p - a_q|$$

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$$\approx |2.177 - 1.074|$$

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&= \epsilon
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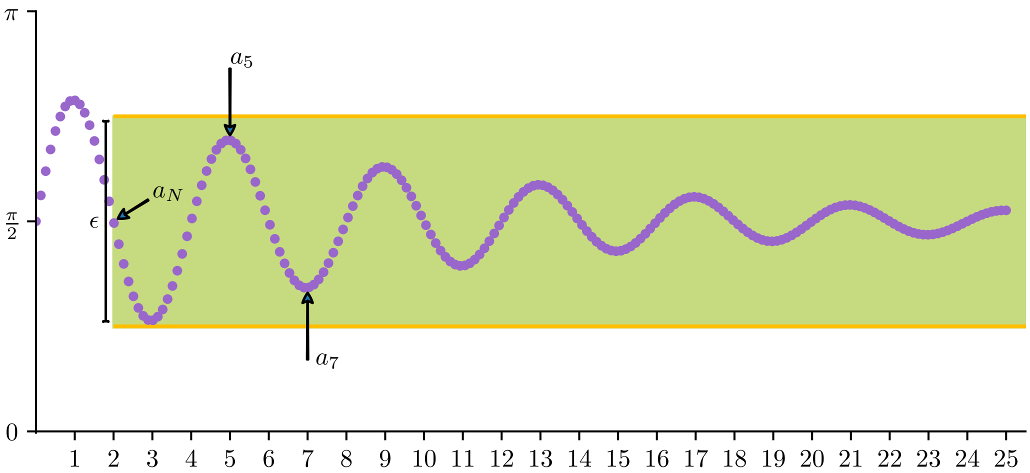
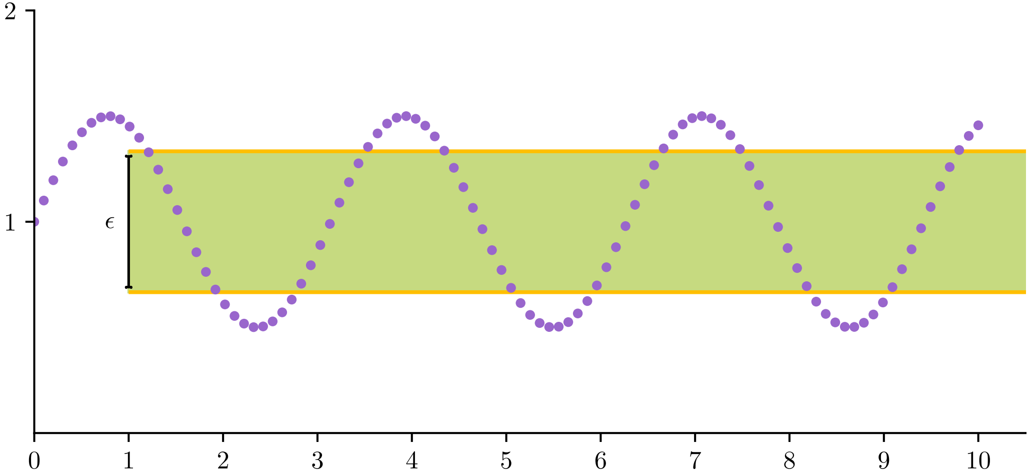


π
π/2
0
ϵ
a_N
a_5
a_7
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25


2
1
ϵ
0
1
2
3
4
5
6
7
8
9
10

think about the sequence

$$\{a_n\} = \{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \dots\}$$

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$$\lim_{n\to\infty} a_n = \pi$$

think about the function $f(x) = x^2$

think about the function $f(x) = x^2$...

but force the domain of f to be the rational numbers $\mathbb{Q}$.

the *intermediate value theorem* says that

on $[a, b]$, $f(x)$ takes on any value between $f(a)$ and $f(b)$ in $[a, b]$.

if $a = 1$ and $b = 2$, then $f(a) = 1$ and $f(b) = 4$...

if $a = 1$ and $b = 2$, then $f(a) = 1$ and $f(b) = 4$...

$$f(a) < \pi < f(b)$$

... but if we take

$$\{f(\sqrt{a_n})\} = \{f(\sqrt{3}), f(\sqrt{3.14}), f(\sqrt{3.1415}$$

$$\lim_{n\to\infty} f(\sqrt{a_n}) = f(\sqrt{\pi}) = \pi$$

$$1 < \pi < 4$$

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$$1 < \pi < 4$$

... but is π a rational number?

the intermediate value theorem *fails.*

the real numbers $\mathbb{R}$ are *complete*:

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every Cauchy sequence converges to a real number.