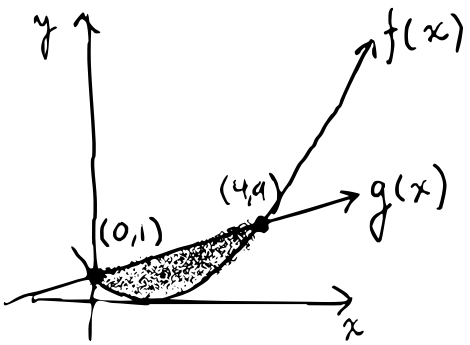
Week 2 Recitation Problems

MATH:114, Recitations 309 and 310

1. Suppose $f(x) = (x-1)^2$ and $g(x) = 2(x-1)+3$. Plot these functions, and shade the region enclosed by them. Where do the curves intersect? What is the area of the enclosed region?



$$(x-1)^2 = 2(x-1)+3$$
$$x^2-2x+1 = 2x-2+3$$
$$x^2-4x = 0$$
$$x(x-4) = 0$$

intersections @ $x=0$, $x=4$

$\Rightarrow (0, f(0)), (4, f(4))$

$\Rightarrow (0,1), (4,9)$

$$\text{area} = \int_0^4 \text{top} - \text{bottom}\, dx$$
$$= \int_0^4 2(x-1)+3-(x-1)^2\, dx$$
$$= \int_0^4 4x - x^2\, dx$$
$$= 2x^2 - \frac{x^3}{3}\Big|_0^4$$
$$= \frac{32}{3}$$

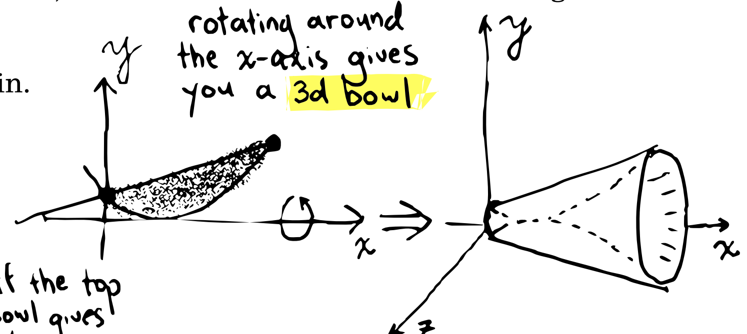
2. Take the region you drew and spin it all the way around the x-axis. This is called a *solid of revolution*, and it lives in 3-dimensional space. Now, draw cross-sections of the solid along the

(a) xy plane that intersects the z-axis at the origin.

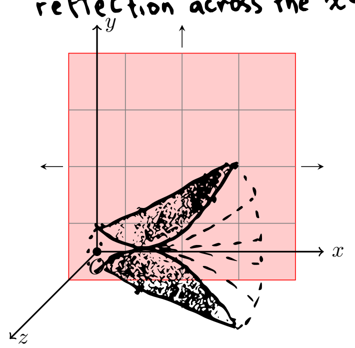
(b) yz plane that intersects the x-axis at $\tilde{x} = 3$

(c) xz plane that intersects the y-axis at $y = 1$.

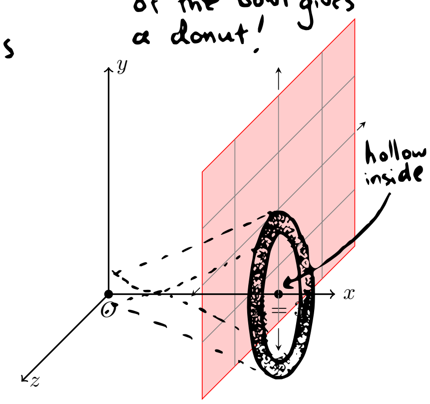
rotating around the x-axis gives you a 3d bowl



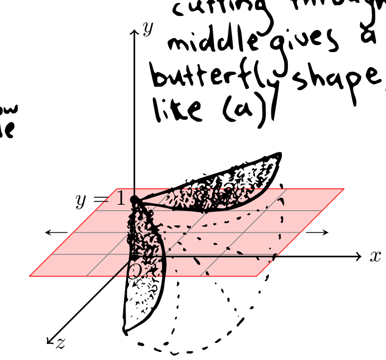
cutting down the center, top to bottom, gives our original area and its reflection across the x-axis



cutting off the top of the bowl gives a donut!



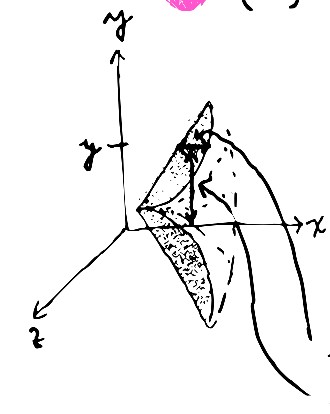
cutting through the middle gives a butterfly shape, like (a)!



- the sum of all the infinitely many, infinitely thin areas is the volume of the solid!

3. Write a formula for the area of each cross-section (hint: one is too hard, and two are familiar geometric ones). If we "shift" our intersection plane along the length of the solid, finding the area of each cross-section, what is the *sum* of these areas?

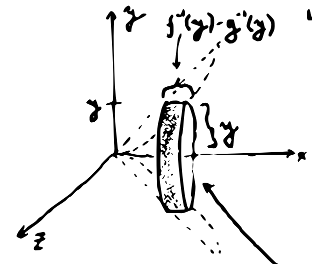
- (a) and (c) are similar — we just need to change our perspective!



slice with the xy plane, intersecting the z-axis at $z=1$. to find the area, we use a geometric idea!

invert f and g, like last week's problem 3: we get two functions f^{-1} and g^{-1} that take y values to x values.

this distance is $f^{-1}(y)-g^{-1}(y)$, and this distance is y.



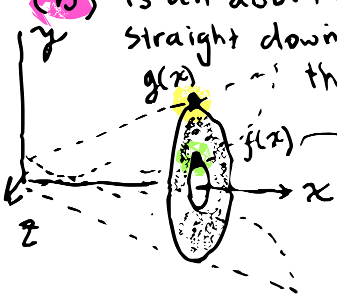
where $x=0$ and $z=0$, we get a line segment that is $f^{-1}(y)-g^{-1}(y)$ long. if we rotate that segment all the way around the x-axis, we get a hollow band, or shell.

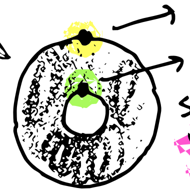
this shell is a hollow cylinder with no top or bottom — but its radius is y and its height is $f^{-1}(y)-g^{-1}(y)$, so its surface area is

$$2\pi r h = 2\pi(y)(f^{-1}(y)-g^{-1}(y)).$$

if we add up the surface areas of all the cylinders (shells), we get the volume!

(b) is all about circles. look at the cross-section straight down the center - it's a washer! to find the area, we need radii.





→ value of top func

→ value of bottom func

so... area of a washer is $\pi(R)^2-\pi(r)^2$

$$=\pi(\text{top function})^2-\pi(\text{bottom fn})^2$$

$$=\pi(g(x))^2-\pi(f(x))^2$$

4. Choose an intersection plane. Write an integral that represents the sum of all the cross-sectional areas we get by shifting the plane along the solid.

to add up all the areas — aka the surface areas of the washers or shells — we use integrals!

(a) and (c) like last week, we set up an integral along the y-axis:

$$V=\int_a^b(\text{area of shell at } y)\,dy$$

$$=\int_a^b 2\pi(y)(f^{-1}(y)-g^{-1}(y))\,dy$$

$$=2\pi\int_a^b y\cdot(f^{-1}(y)-g^{-1}(y))\,dy.$$

the shell method!

(b) like last week, we set up an integral on the x-axis using diffs of functions:

$$V=\int_a^b\pi(\text{top fn})^2-\pi(\text{bottom fn})^2\,dx$$

$$=\pi\int_a^b g(x)^2-f(x)^2\,dx$$

the washer method!

note: the axis of revolution determines the axis of integration! if we revolved the area in (1) around the y axis instead of the x axis, which axis of integration would we use for each method?

additionally, be careful constructing inverse functions!