Generalized cluster algorithms for Potts lattice gauge theory

Anthony E. Pizzimenti @ JMM · January 7th, 2026

Ben Schweinhart
George Mason University

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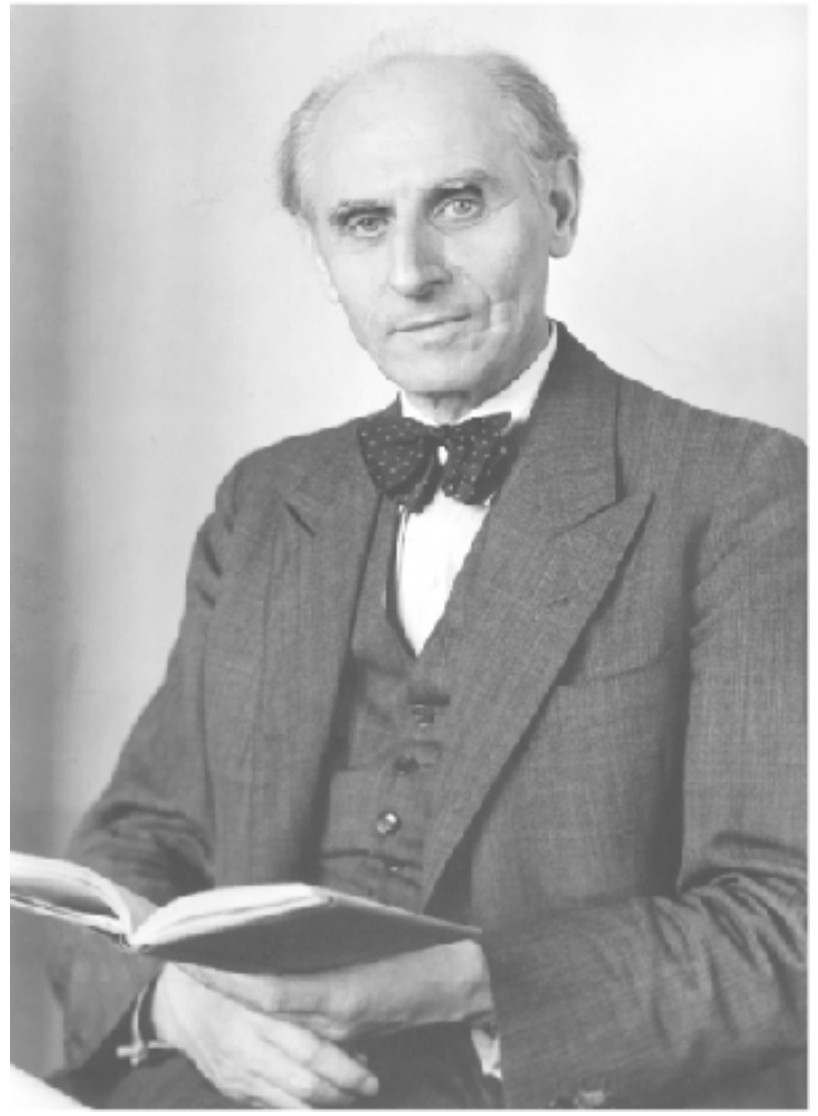
the plan

1. classical models
2. homological generalization
3. practicalities and future work

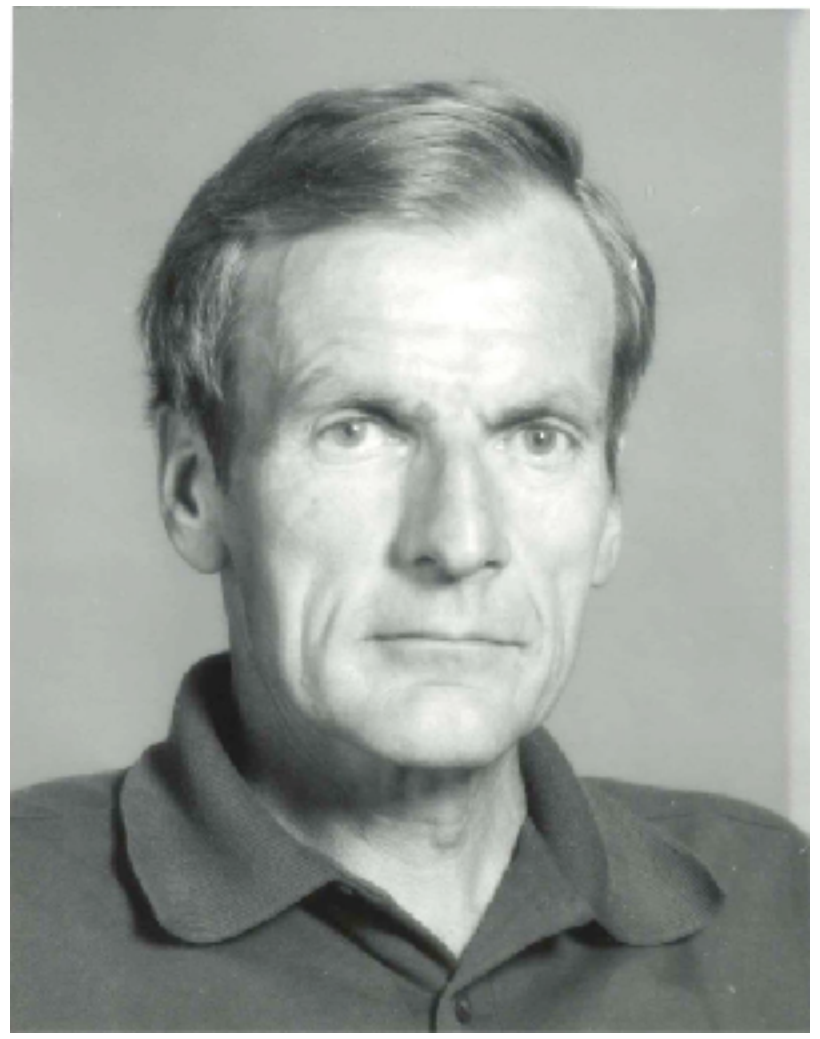
1. classical models

Ising/Potts models

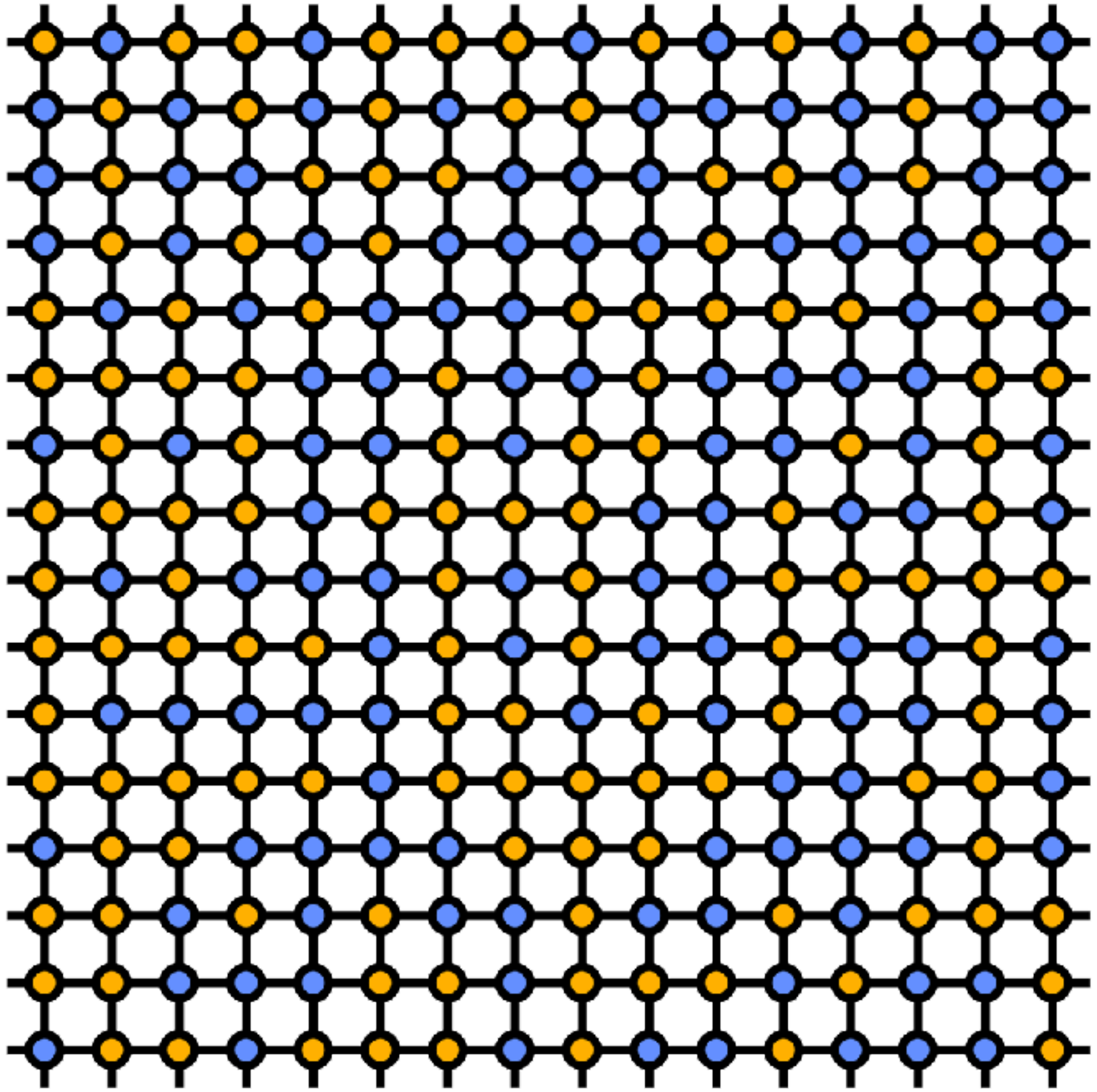
Ernst Ising
c. University of Cologne



Wilhelm Lenz
c. Universität Rostock

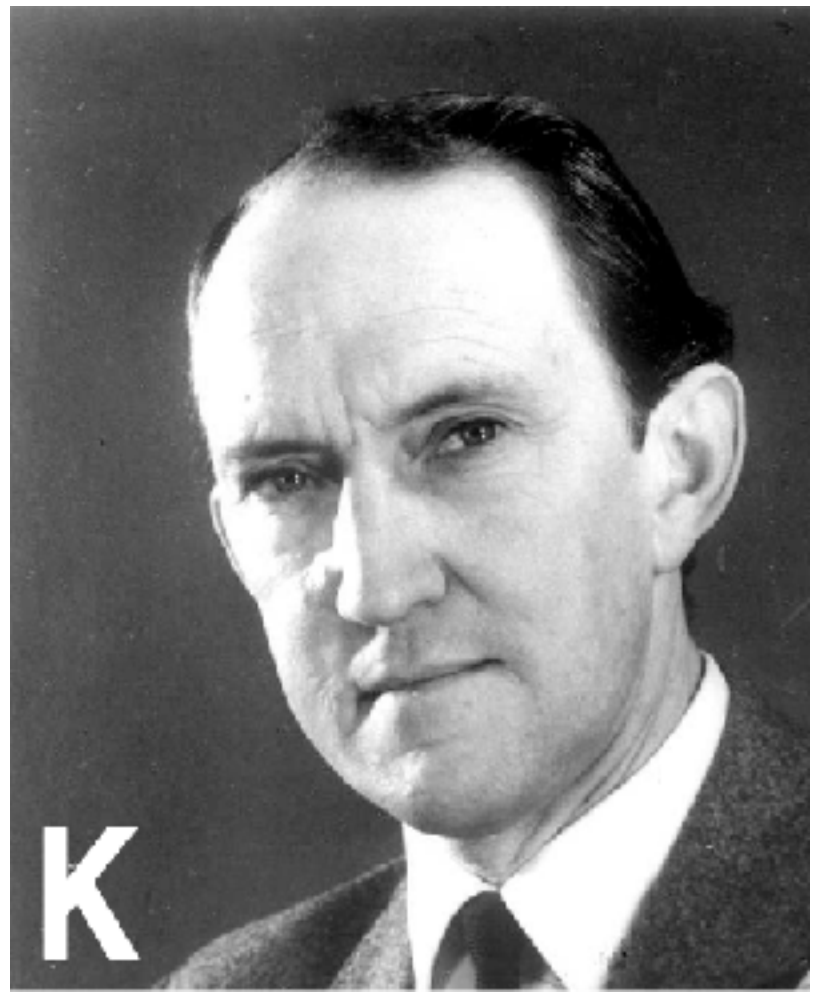


Renfrey Potts
c. University of Adelaide



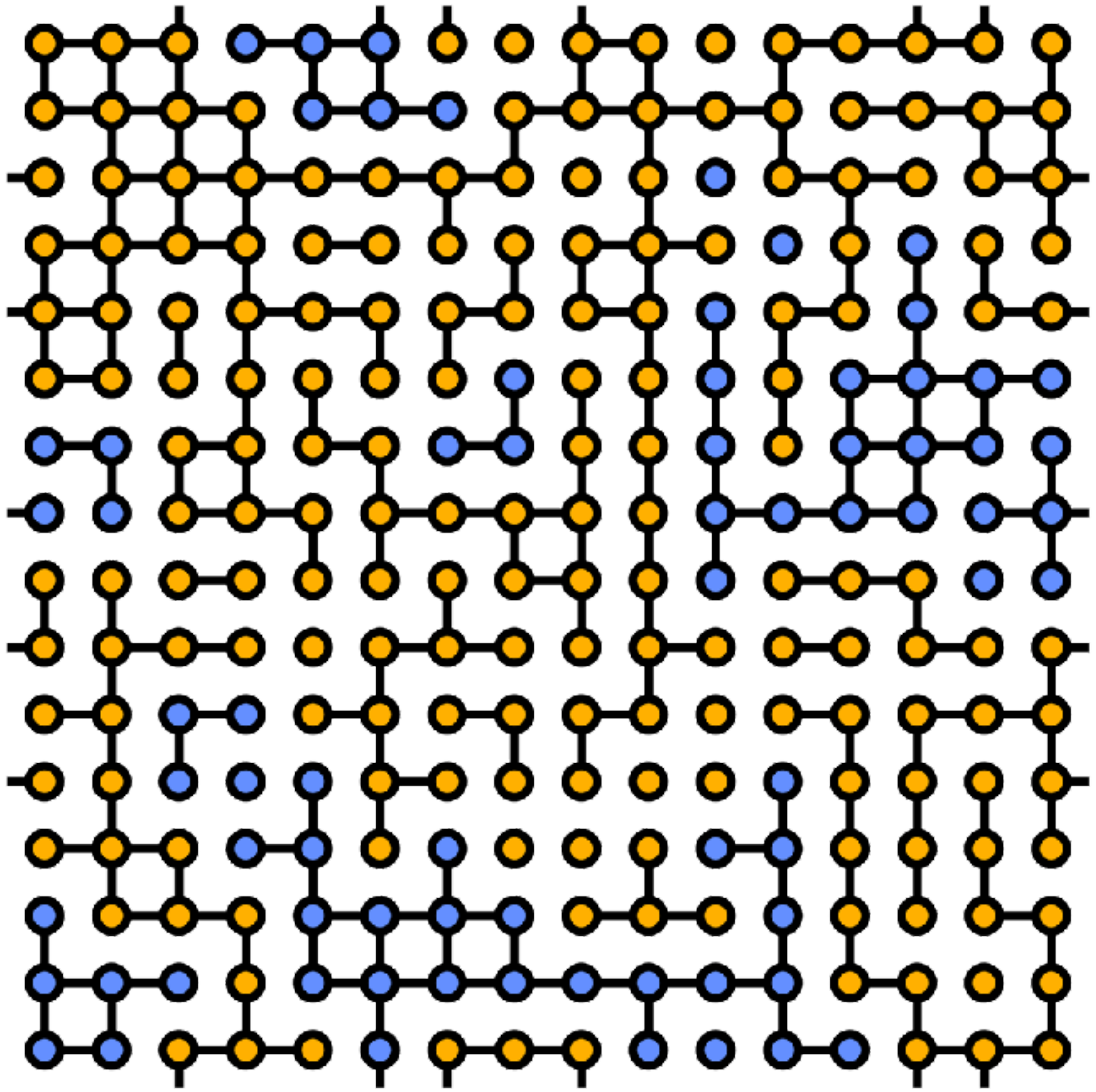
$$\text{Potts}(f) \propto e^{-\beta \mathbf{H}(f)}$$

Kees Fortuin
c. The Random-Cluster Model, *Geoffrey Grimmett*



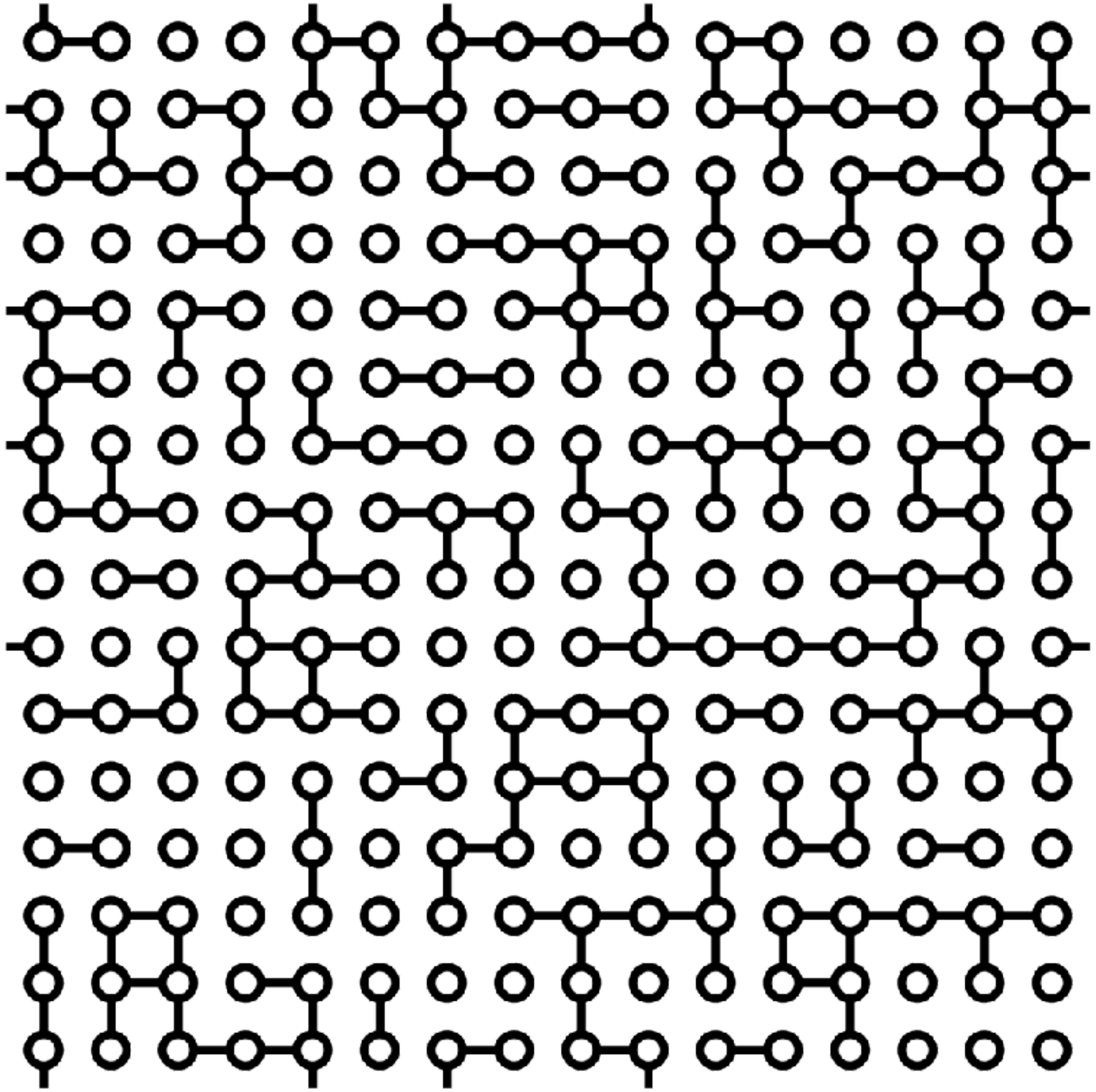
Piet Kasteleyn
c. The Random-Cluster Model, *Geoffrey Grimmett*

FK random-cluster model

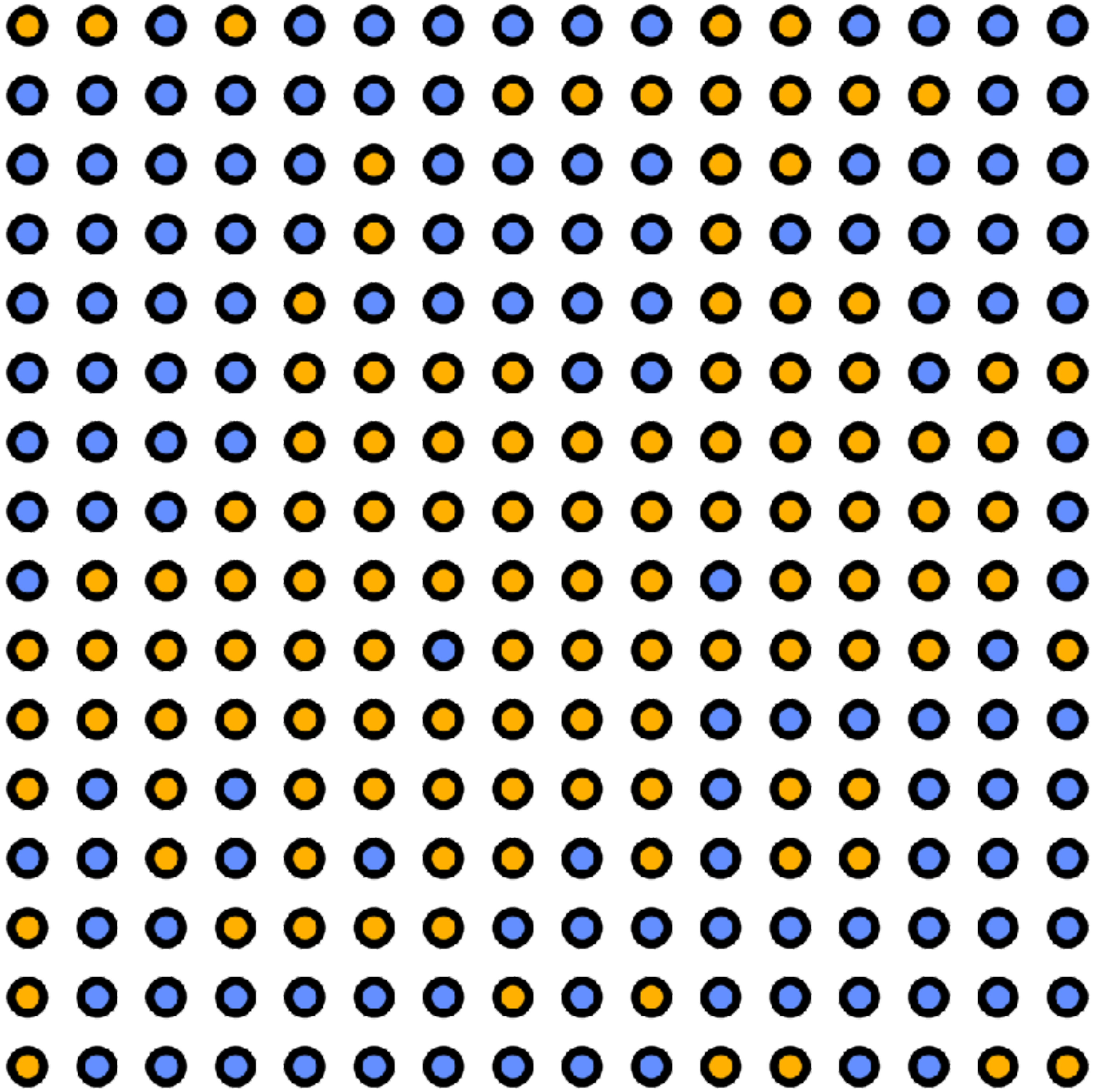


$$\mathrm{RCM}(Q) \propto p^{|Q|}\ (1-p)^{|G|-|Q|}\ q^{(\text{\# of components})}$$

Q

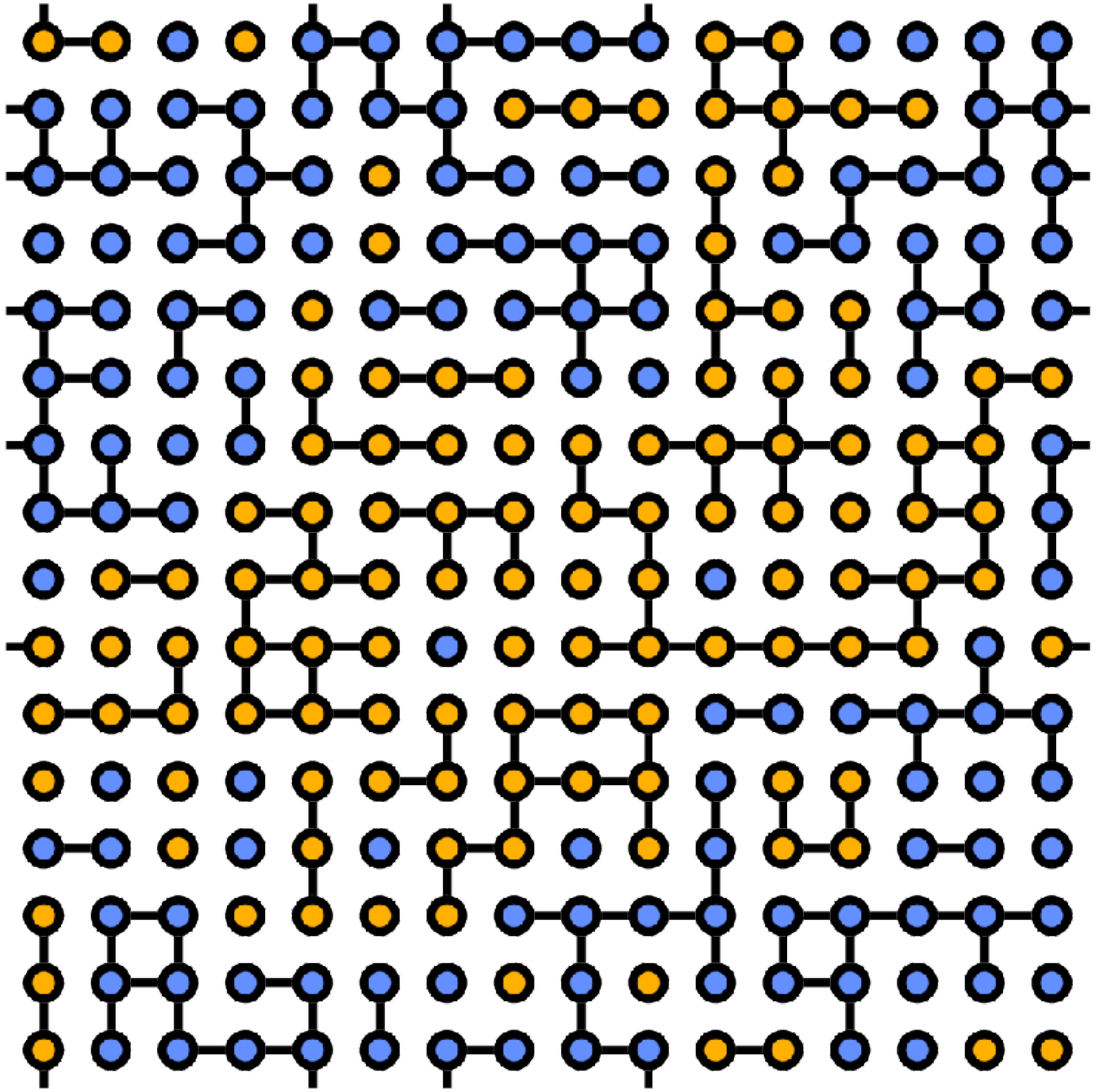


f



(Q, f) is a ***compatible pair***

$(Q, f) \in Z^0(Q; \mathbb{Z}_q)$



f is constant on each component of Q

$$|Z^0(Q; \mathbb{Z}_q)| = q^{(\text{\# of components})}$$

\# of compatible pairs $(Q, -)$

$\mathbf{H}(f) = -$(# of edges with agreeing spins)

$$\text{Potts}(f) = \frac{e^{-\beta \mathbf{H}(f)}}{\mathcal{Z}(q, \beta)}$$

$$\mathcal{Z}(q, \beta) = \sum_{f} e^{-\beta \mathbf{H}(f)}$$

the event that f is compatible with the edge $x = (u, v)$

setting $p = 1 - e^{-\beta}$ and $F(f, x) = \{f(u) = f(v)\}$,

$$e^{\beta \mathbf{1}_{F(f,x)}} = e^{\beta}\left(p\mathbf{1}_{F(f,x)} + (1-p)\right)$$

$\mathbf{1}_{F(f,x)} = 1$ when $F(f, x)$ holds, 0 otherwise

$$\sum_f e^{-\beta \mathbf{H}(f)} = \sum_f \prod_x e^{\beta \mathbf{1}_{F(f,x)}}$$

$$e^{\beta \mathbf{1}_{F(f,x)}} = e^{\beta}\left(p\mathbf{1}_{F(f,x)} + (1-p)\right)$$

$$\sum_f e^{-\beta \mathbf{H}(f)} = \sum_f \prod_x e^{\beta \mathbf{1}_{F(f,x)}}$$

$$\mathscr{Z}(q,\beta) = \sum_f e^{-\beta \mathbf{H}(f)}$$

$$= \sum_f \prod_x e^{\beta}\left(p\mathbf{1}_{F(f,x)} + (1-p)\right)$$

$$= e^{\beta |G|}\left[\sum_Q p^{|Q|}\,(1-p)^{|G|-|Q|}\,|Z^0(Q;\mathbb{Z}_q)|\right]$$

# of compatible pairs $(Q, -)$ → $|Z^0(Q;\mathbb{Z}_q)|$

$\mathrm{RCM}(Q)$ → $p^{|Q|}(1-p)^{|G|-|Q|}$

# of compatible pairs $(Q, -)$

$$\mathrm{RCM}(Q) = \frac{p^{|Q|}\,(1-p)^{|G|-|Q|}\,|Z^0(Q;\mathbb{Z}_q)|}{e^{-\beta|G|}\,\mathcal{Z}(q,\beta)}$$

Robert G. **E**dwards

$$\mathrm{ES}(f, Q) \propto \prod_{x} \left[(1-p)\mathbf{1}_{x \notin Q} + (p)\mathbf{1}_{x \in Q}\mathbf{1}_{F(f,x)} \right]$$

Alan D. **S**okal

$$\mathrm{ES}(f, -) = \sum_{Q} \mathrm{ES}(f, Q) = \mathrm{Potts}(f)$$

$$\mathrm{ES}(-, Q) = \sum_{f} \mathrm{ES}(f, Q) = \mathrm{RCM}(Q)$$

$$\mathrm{ES}(f \mid Q)$$

uniform over f constant on components of Q

$$\mathrm{ES}(Q \mid f)$$

independent percolation on edges compatible with f

how do we sample from these distributions?

→ ***single-spin Glauber dynamics***

Swendsen-Wang dynamics

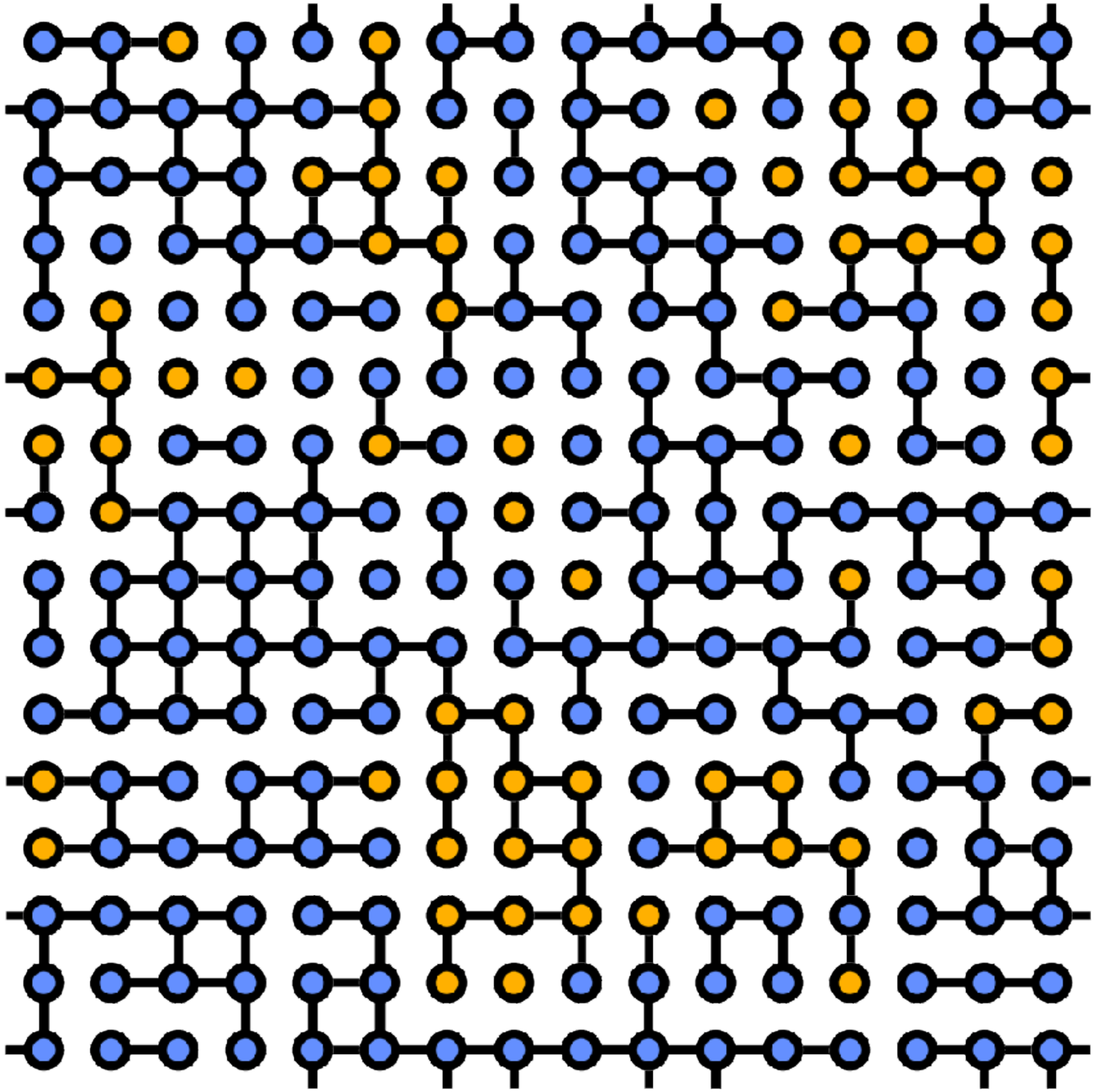
Swendsen-Wang dynamics

(1) Given compatible Q_t and f_t,

(2) Sample Q_{t+1} from $\mathrm{ES}(-\mid f_t)$ (independent percolation over edges with matching spins).

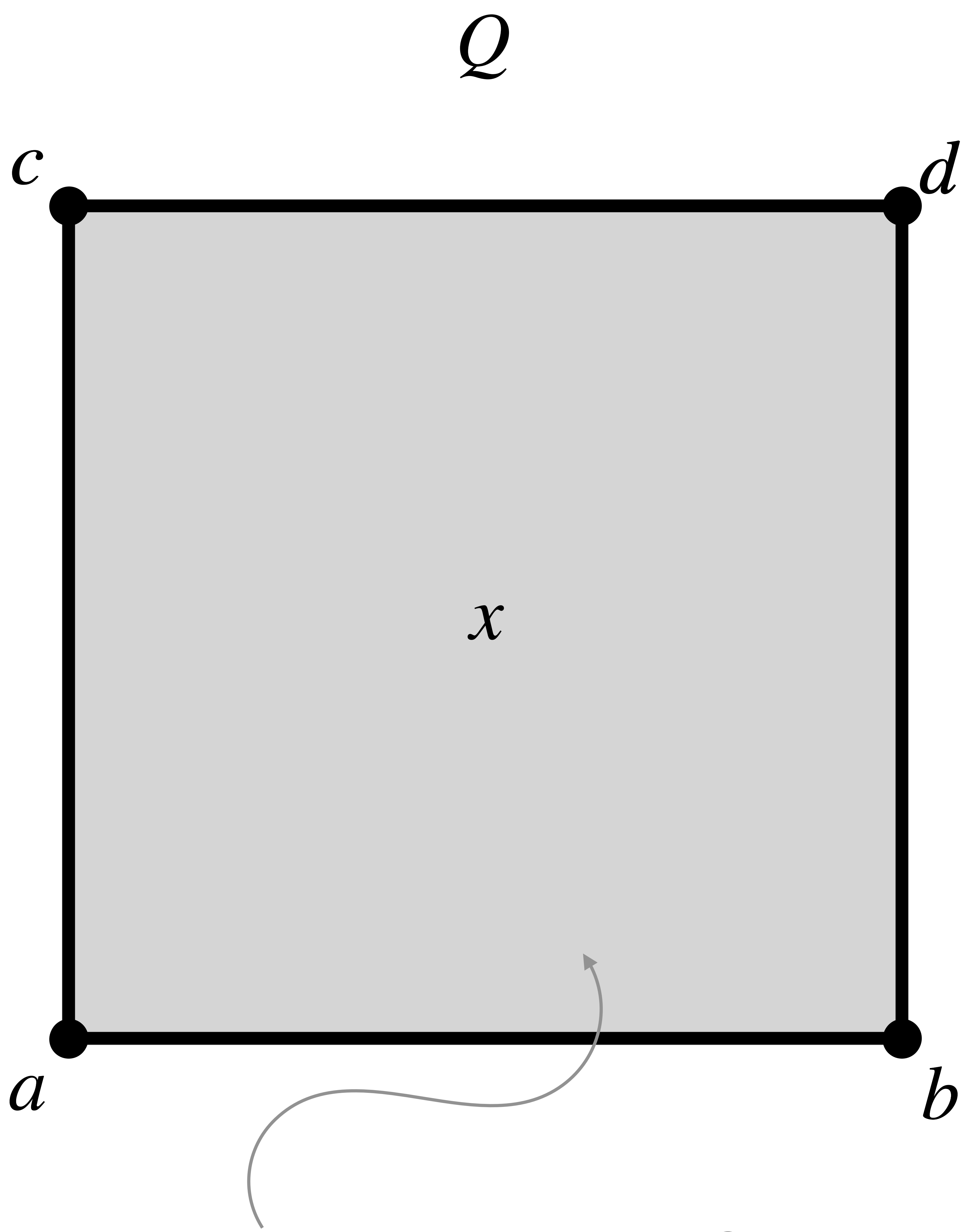
(3) Sample f_{t+1} from $\mathrm{ES}(-\mid Q_{t+1})$ (uniform random over spin configurations constant on components).

(4) Set $t := t+1$ and return to Step (1).

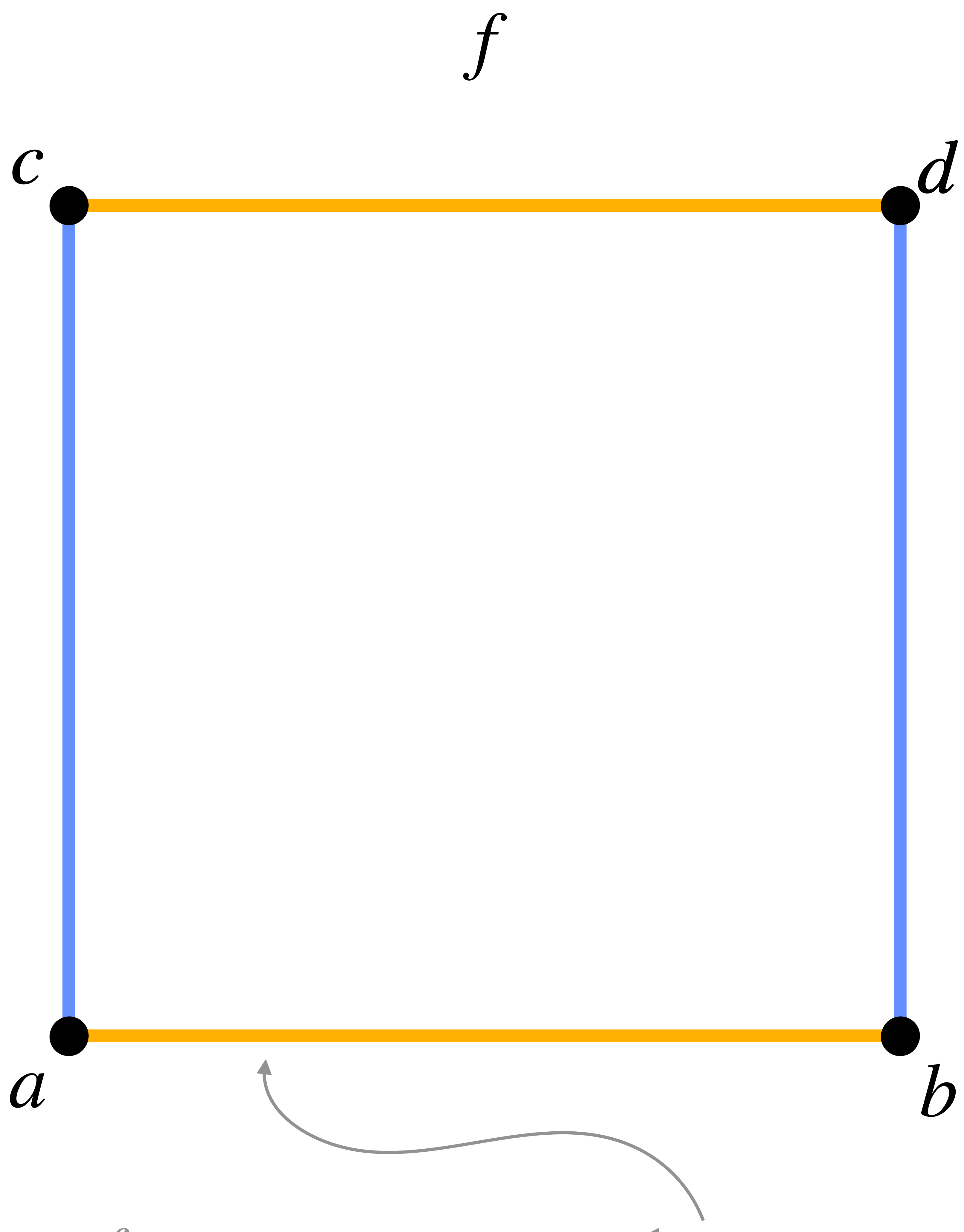


… what if we want to put spins on ***edges*** instead?

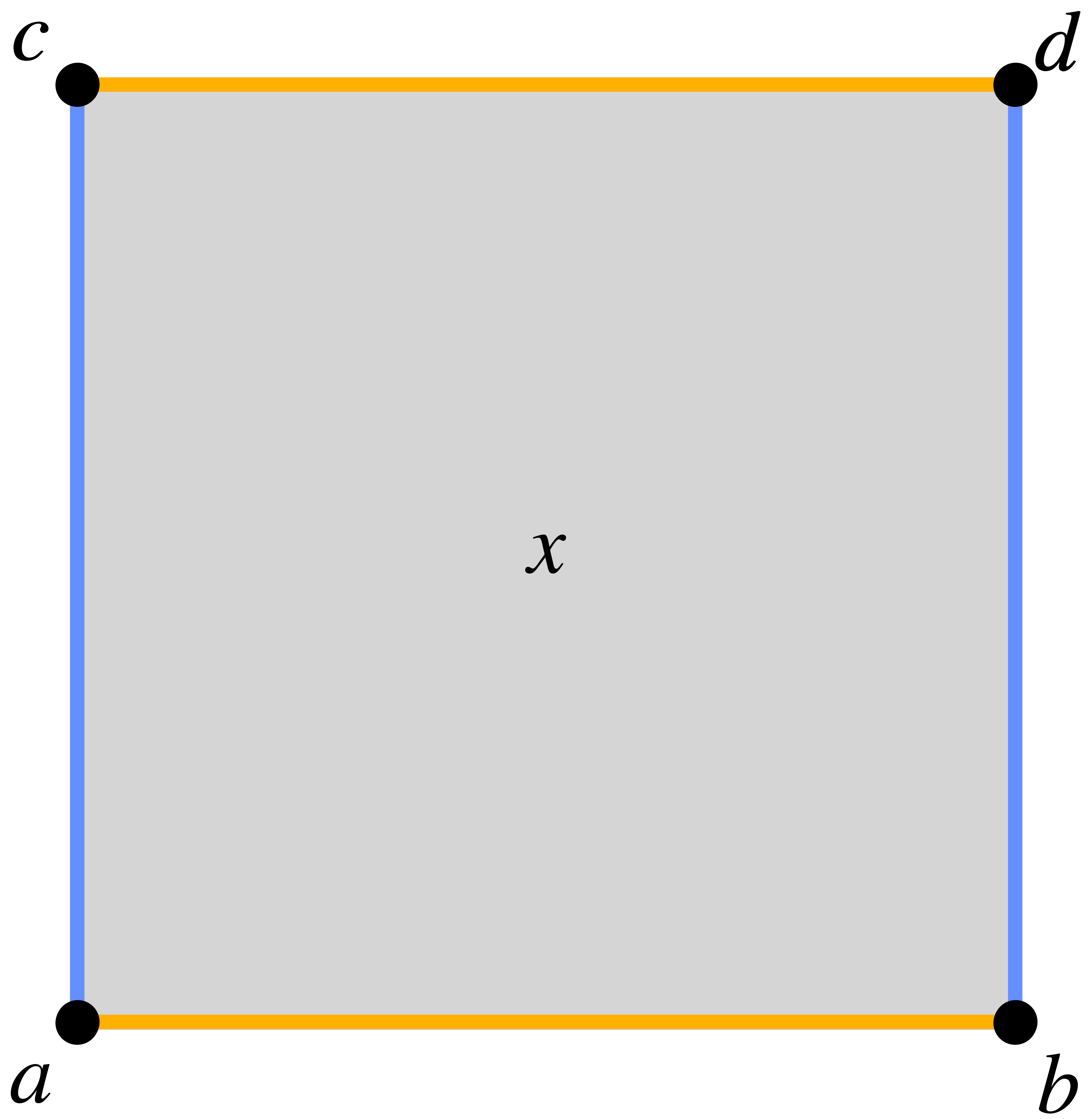
2. homological generalization

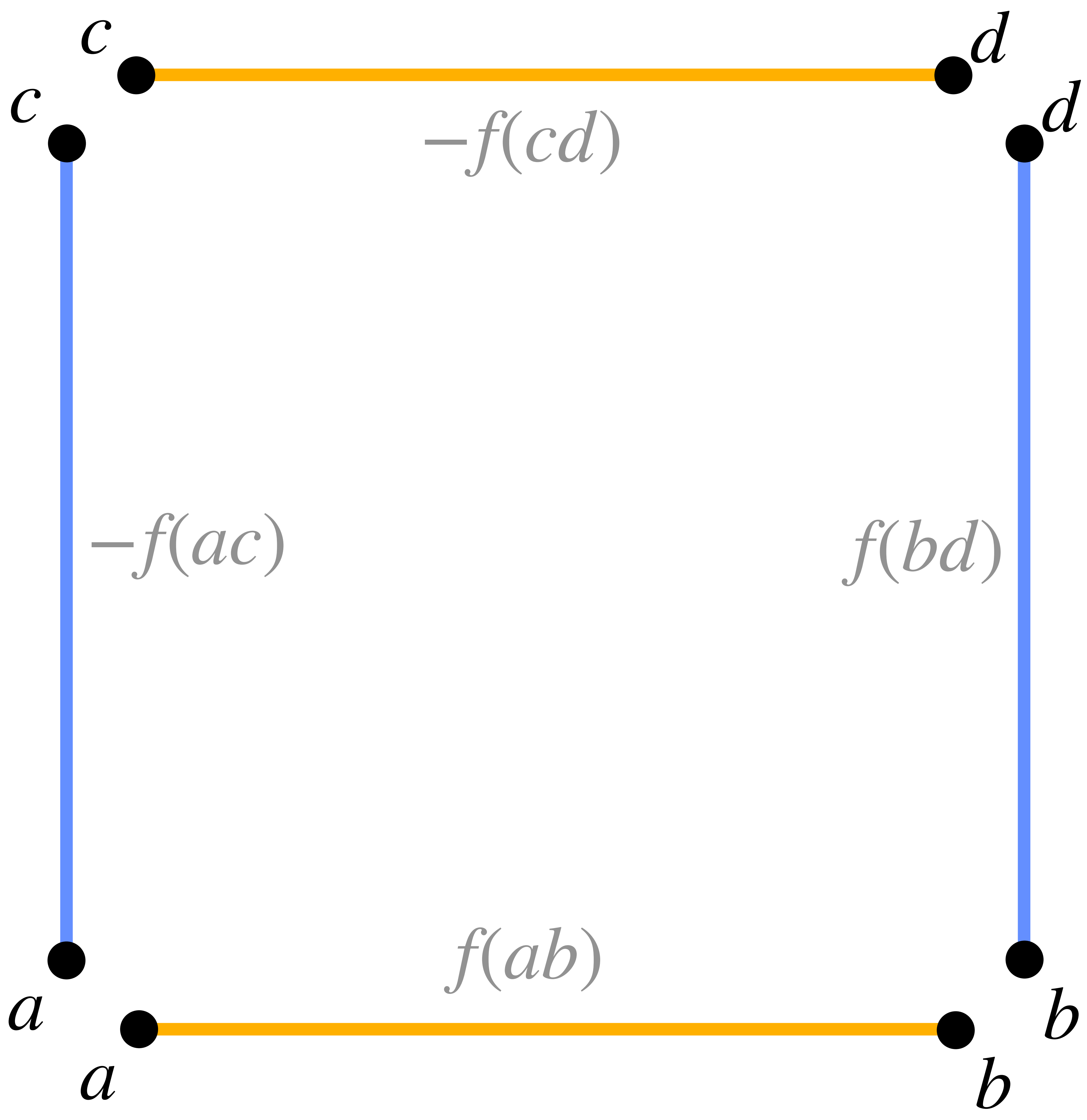


x is a ***plaquette*** (or 2-***cell***)



f is a ***configuration*** (or 1-***cochain)***


c
d
x
a
b



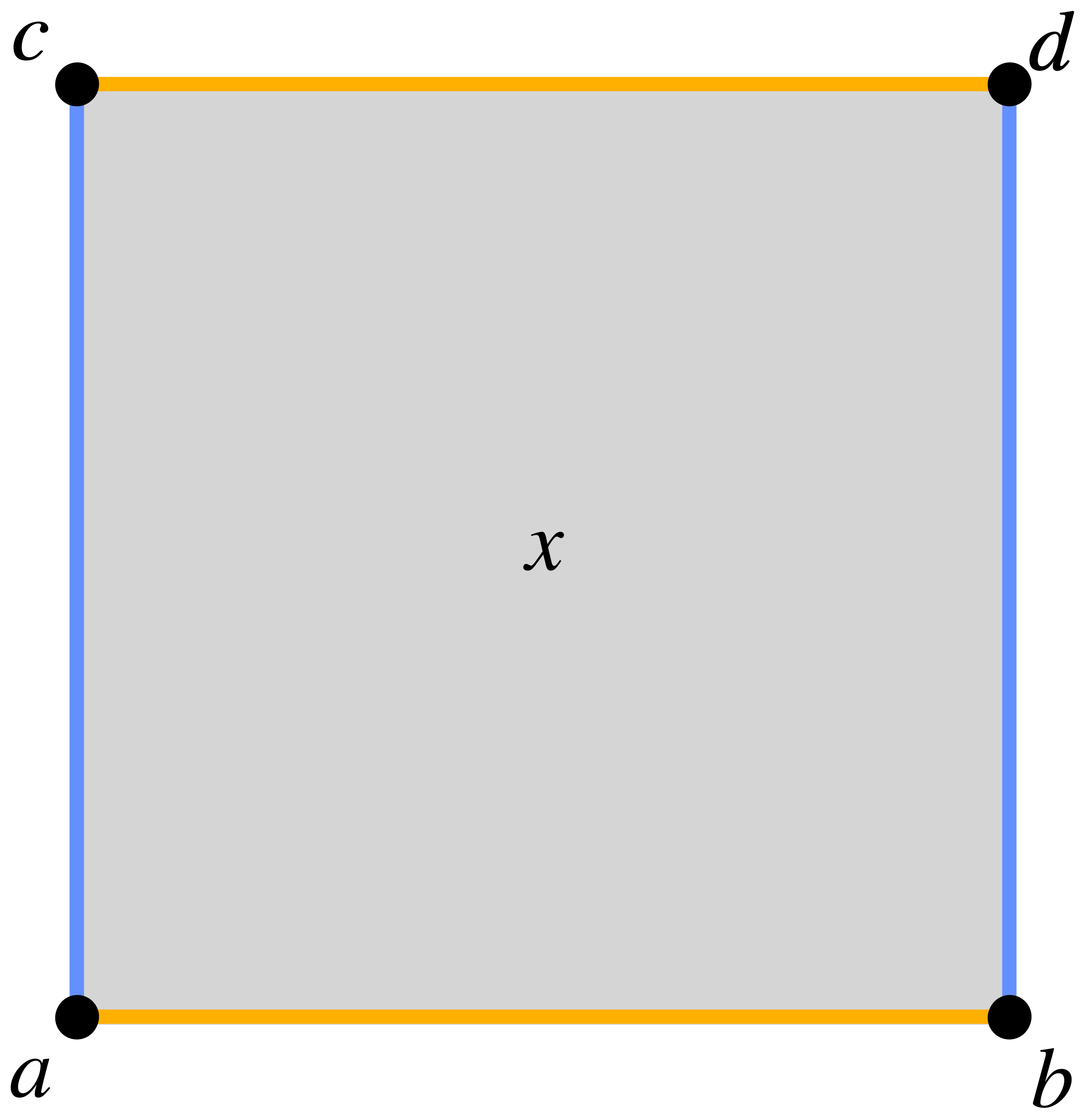
the ***coboundary*** is the oriented sum of spins on the boundary of x

$$
\begin{aligned}
(\delta^1 f)(x) &= f(\partial_2(x)) \\
&= f(ab) - f(ac) + f(bd) - f(cd) \\
&= 1 - 0 + 0 - 1 \\
&= 0
\end{aligned}
$$

x is ***nonfrustrated*** if $(\delta^1 f)(x) = 0$, and ***frustrated*** otherwise

(Q, f) is a ***compatible pair***

$(Q, f) \in Z^1(Q; \mathbb{Z}_q)$



$(\delta^1 f)(x) = 0$ for all plaquettes $x \in Q$

Potts lattice gauge theory (PLGT)

$\mathbf{H}(f) := -\sum_x \mathbf{1}_{(\delta^1 f)(x)=0}$ counts nonfrustrated plaquettes

$$\mathrm{PLGT}(f) = \frac{e^{-\beta \mathbf{H}(f)}}{\mathscr{Z}(q,\beta)}$$

$$\mathscr{Z}(q,\beta) = \sum_f e^{-\beta \mathbf{H}(f)}$$

$$e^{\beta \mathbf{1}_{F(f,x)}} = e^{\beta}\left(p\mathbf{1}_{F(f,x)} + (1-p)\right)$$

$$\sum_f e^{-\beta \mathbf{H}(f)} = \sum_f \prod_x e^{\beta \mathbf{1}_{F(f,x)}}$$

$$\mathcal{Z}(q,\beta) = \sum_f e^{-\beta \mathbf{H}(f)}$$

$F(f,x) = \{\ x$ is nonfrustrated wrt $f\ \}$

$$= \sum_f \prod_x e^{\beta}\left(p\mathbf{1}_{F(f,x)} + (1-p)\right)$$

# of compatible pairs $= c(X)\,|\,H^1(Q;\mathbb{Z}_q)\,|$

$$= e^{\beta|X|}\left[\sum_Q p^{|Q|}\,(1-p)^{|X|-|Q|}\,c(X)\,|\,H^1(Q;\mathbb{Z}_q)\,|\right]$$

of ***gauge transformations*** (aka the size of the coboundary group $B^1(Q;\mathbb{Z}_q)$)

cohomology of Q

plaquette random-cluster model (PRCM)

$$\mathrm{PRCM}(Q) = \frac{p^{|Q|}\,(1-p)^{|X|-|Q|}\,\left|H^1(Q;\mathbb{Z}_q)\right|}{\frac{e^{-\beta|X|}}{c(X)}\,\mathcal{Z}(q,\beta)}$$

$$\mathrm{ES}(f, -) = \sum_{Q} \mathrm{ES}(f, Q)$$
$$= \mathrm{PLGT}(f)$$

$$\mathrm{ES}(-, Q) = \sum_{f} \mathrm{ES}(f, Q)$$
$$= \mathrm{PRCM}(Q)$$

$$\mathrm{ES}(f \mid Q)$$

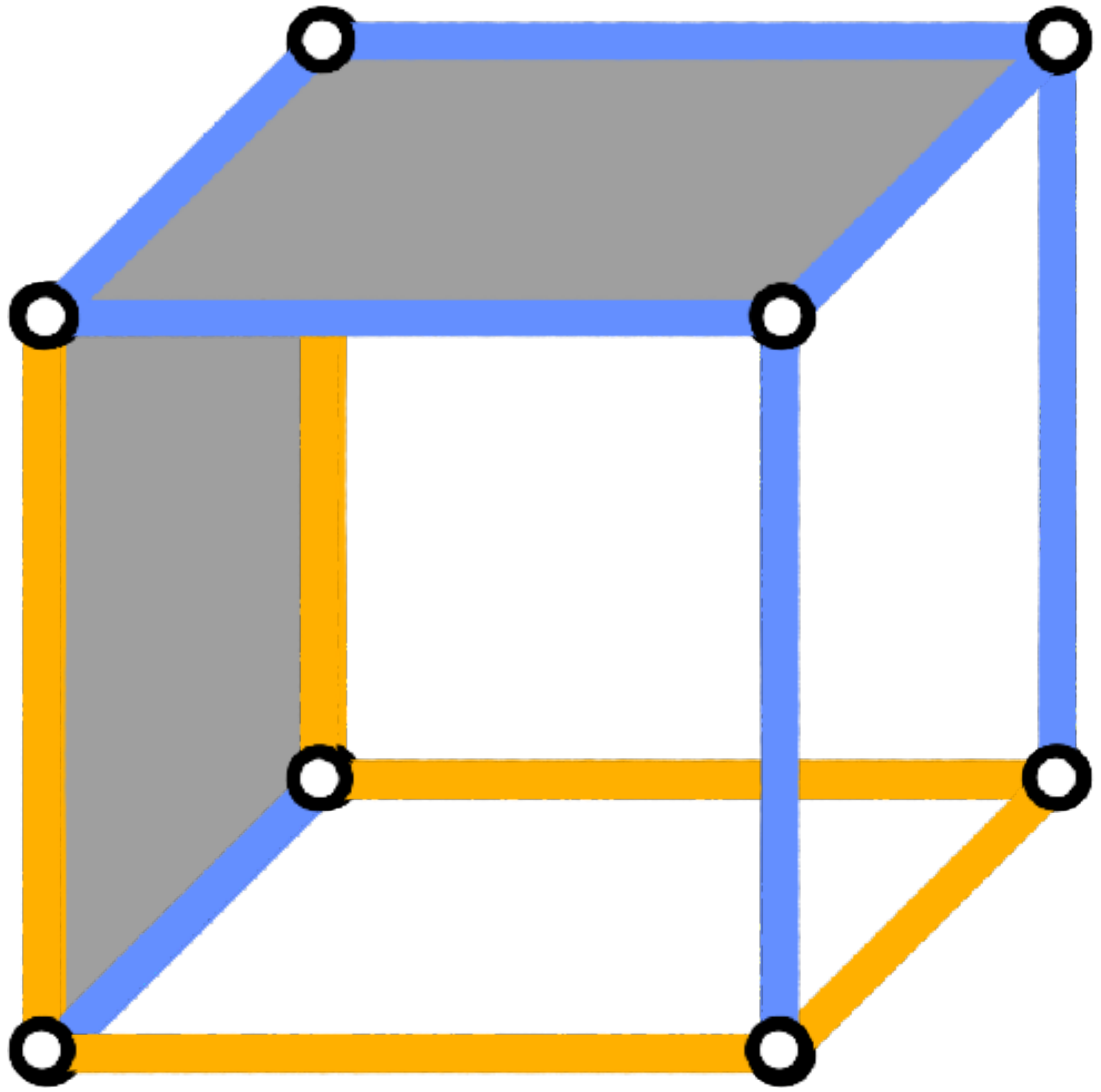
uniform over configurations f compatible with Q

$$\mathrm{ES}(Q \mid f)$$

independent percolation on nonfrustrated plaquettes wrt f

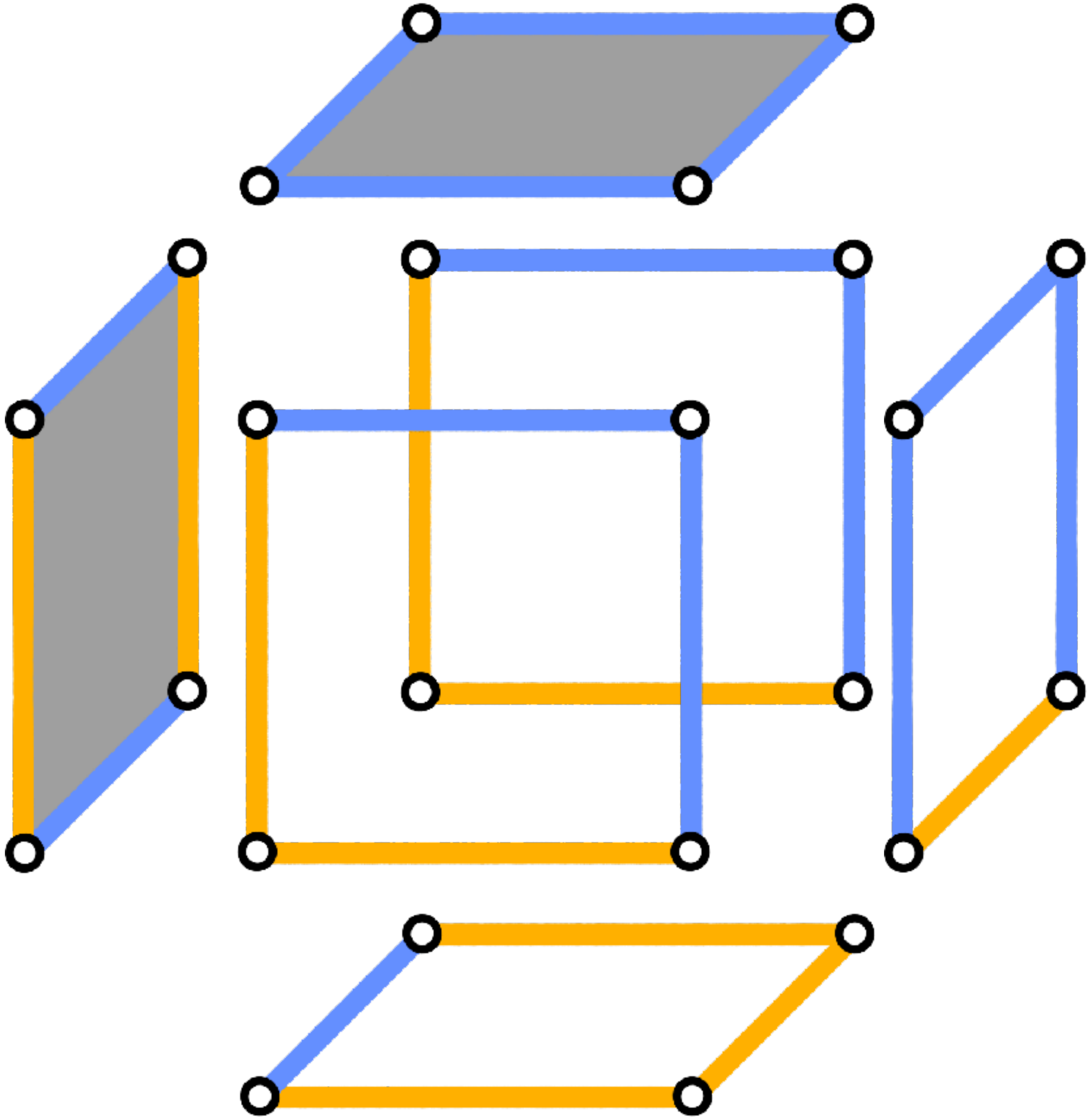
Swendsen-Wang dynamics

(1) Given Q_t and f_t,



Swendsen-Wang dynamics

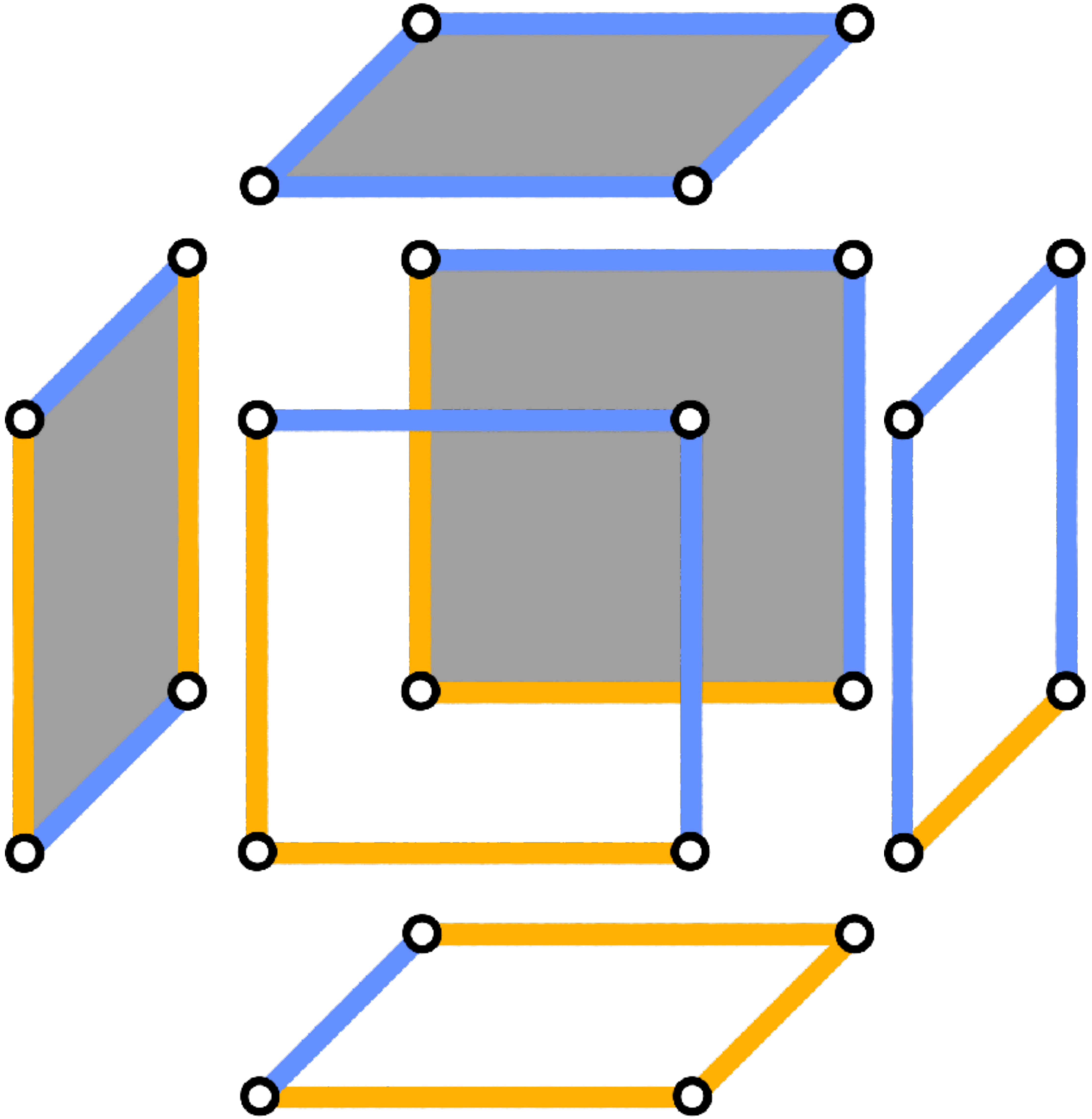
(1) Given Q_t and f_t,



Swendsen-Wang dynamics

(1) Given Q_t and f_t,

(2) Sample Q_{t+1} from $\mathrm{ES}(-\mid f_t)$ (independent percolation over nonfrustrated plaquettes).

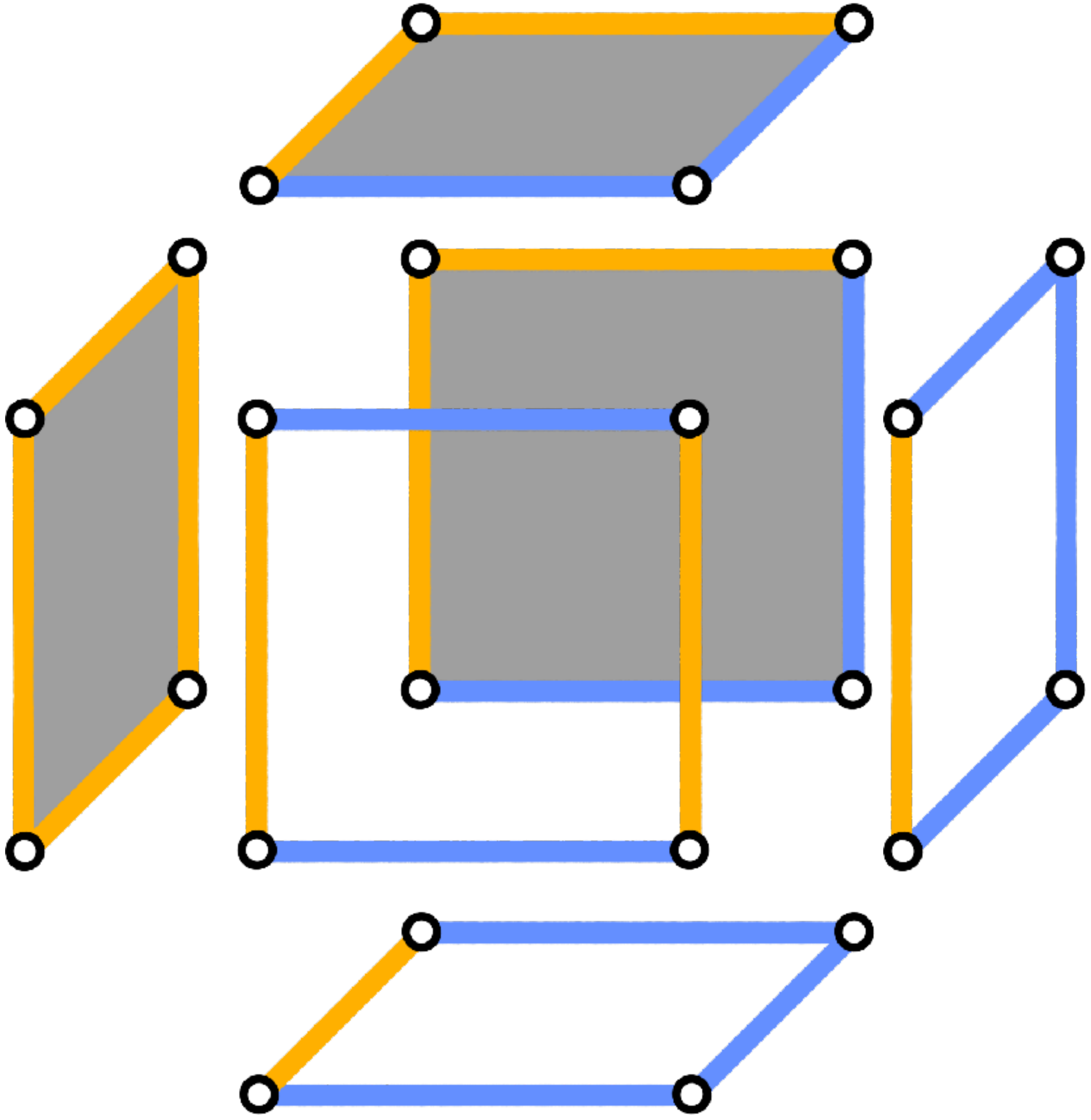


Swendsen-Wang dynamics

(1) Given Q_t and f_t,

(2) Sample Q_{t+1} from $\mathrm{ES}(-\mid f_t)$ (independent percolation over nonfrustrated plaquettes).

(3) Sample f_{t+1} from $\mathrm{ES}(-\mid Q_{t+1})$ (uniform random over configurations compatible with Q_{t+1}).



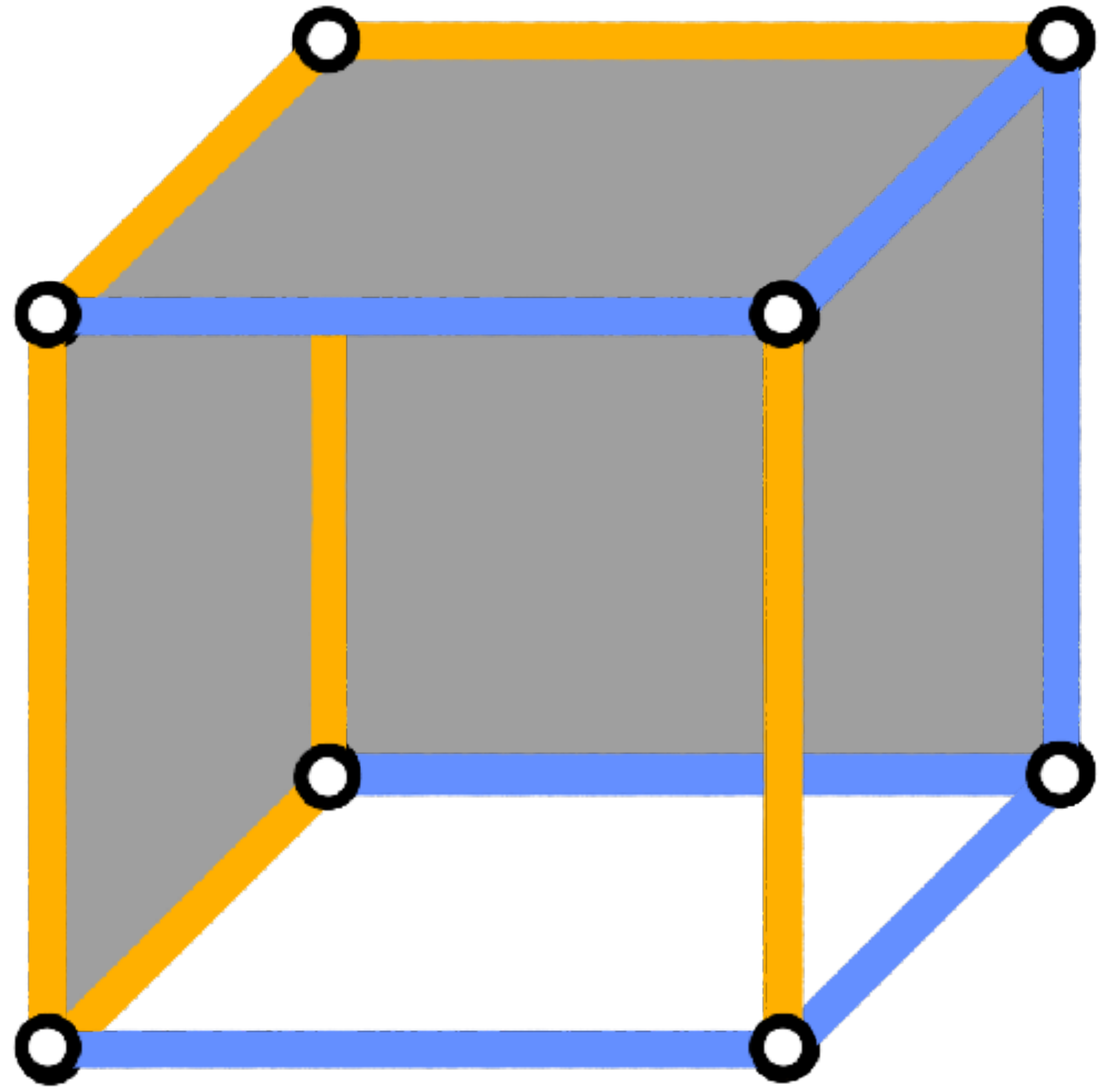
Swendsen-Wang dynamics

(1) Given Q_t and f_t,

(2) Sample Q_{t+1} from $\mathrm{ES}(\,-\mid f_t)$ (independent percolation over nonfrustrated plaquettes).

(3) Sample f_{t+1} from $\mathrm{ES}(\,-\mid Q_{t+1})$ (uniform random over configurations compatible with Q_{t+1}).

(4) Set $t := t + 1$ and return to Step (1).

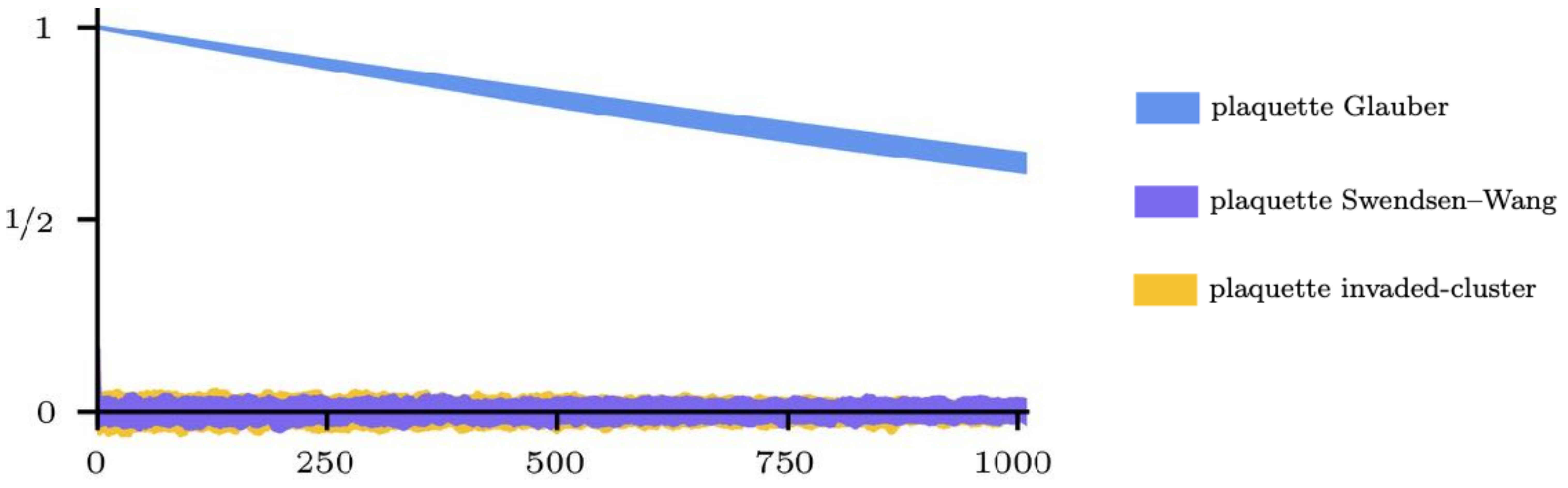


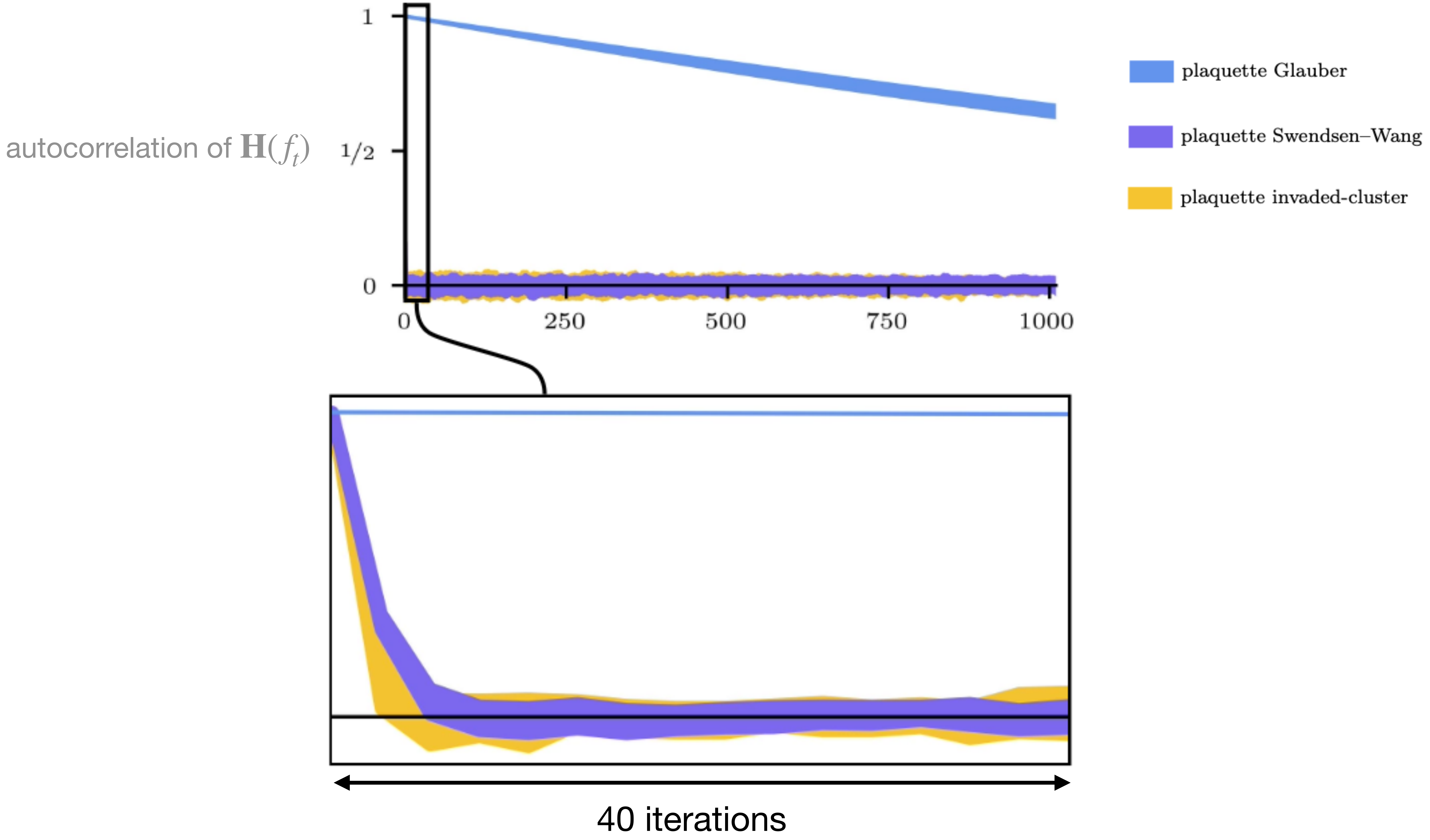
3. practicalities and future work

*“**A**lgebraic **T**opology-**E**nabled **A**lgorith**M**s for **S**pin systems”*

our software, ***ATEAMS***, can simulate all these models in arbitrary dimensions.

autocorrelation of $\mathbf{H}(f_t)$





autocorrelation of $\mathbf{H}(f_t)$
1
1/2
0
0
250
500
750
1000
plaquette Glauber
plaquette Swendsen–Wang
plaquette invaded-cluster
40 iterations

L	**Plaq**	**Field**	**s/it**
10	60,000	$\mathbb{Z}_2$	0.206
		$\mathbb{Z}_3$	0.132
		$\mathbb{Z}_5$	0.093
		$\mathbb{Z}_7$	0.091
20	960,000	$\mathbb{Z}_2$	3.161
		$\mathbb{Z}_3$	1.683
		$\mathbb{Z}_5$	1.076
		$\mathbb{Z}_7$	0.888
30	4,860,000	$\mathbb{Z}_2$	28.550
		$\mathbb{Z}_3$	12.539
		$\mathbb{Z}_5$	6.381
		$\mathbb{Z}_7$	5.234
40	15,360,000	$\mathbb{Z}_2$	224.594
		$\mathbb{Z}_3$	79.565
		$\mathbb{Z}_5$	27.017
		$\mathbb{Z}_7$	19.159

30	4,860,000		
		$\mathbb{Z}_5$	6.381
		$\mathbb{Z}_7$	5.234
40	15,360,000	$\mathbb{Z}_2$	224.594
		$\mathbb{Z}_3$	79.565
		$\mathbb{Z}_5$	27.017
		$\mathbb{Z}_7$	19.159

of ***occupied cells***   (negative) ***total energy***

L	τ_N	τ_N [28]	$\tau_\mathcal{E}$	$\tau_\mathcal{E}$ [28]
4	2.0096 ± 0.2742	2.1169 ± 0.0018	2.3419 ± 0.3041	2.3697 ± 0.0021
6	2.4980 ± 0.3181	2.7257 ± 0.0026	2.8717 ± 0.3529	3.0618 ± 0.0031
8	3.0887 ± 0.3735	3.2298 ± 0.0033	3.5027 ± 0.4125	3.6496 ± 0.0040
12	3.7788 ± 0.4355	4.0638 ± 0.0047	4.2931 ± 0.4783	4.6314 ± 0.0058
16	4.2462 ± 0.4757	4.7701 ± 0.0027	4.8360 ± 0.5214	5.4588 ± 0.0033
24	5.3386 ± 0.5612	5.9567 ± 0.0083	6.3495 ± 0.6384	6.8408 ± 0.0102
32	6.5786 ± 0.6560	6.9303 ± 0.0074	7.4935 ± 0.7220	7.9625 ± 0.0090
48	7.4909 ± 0.7219	8.5612 ± 0.0073	8.7650 ± 0.8142	9.8308 ± 0.0090

future work

1. running even larger systems!
2. computing *dynamical critical exponents*
3. further optimizing *ATEAMS*

thank you!

preprint arxiv.org/abs/2507.13503

software github.com/apizzimenti/ATEAMS

me mason.gmu.edu/~apizzime