Generalized cluster algorithms for Potts lattice gauge theory [arxiv:2507.13503]

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A crash course in probability

The q***-state,*** $(i-1)$***-dim Potts lattice gauge theory (PLGT)*** on a d-dim cubical complex X [1–3] is the family of probability distributions $\nu_{\beta,q}(f) \propto e^{-\beta \mathbf{H}(f)}$ over $(i-1)$***-cochains*** f where

$$\mathbf{H}(f) = \sum_{x \in X^i} \mathbf{1}\left((\delta^{i-1} f)(x) = 0\right).$$

The i***-dim plaquette random-cluster model (PRCM)*** $\mu_{p,q}$ on X [4, 5] is the random subcomplex $P \subseteq X^i$ with distribution

$$\mu_{p,q}(P) \propto p^{|P|}\,(1-p)^{|X^i|-|P|}\left|H^{i-1}(P;\mathbb{Z}/q\mathbb{Z})\right|.$$

> When $i=1$, ***PLGT is the*** q***-state Potts model*** and ***PRCM is the*** q***-state random-cluster model***.

For some intuition, ***Figure 1*** shows a low-dim example of PLGT and PRCM.

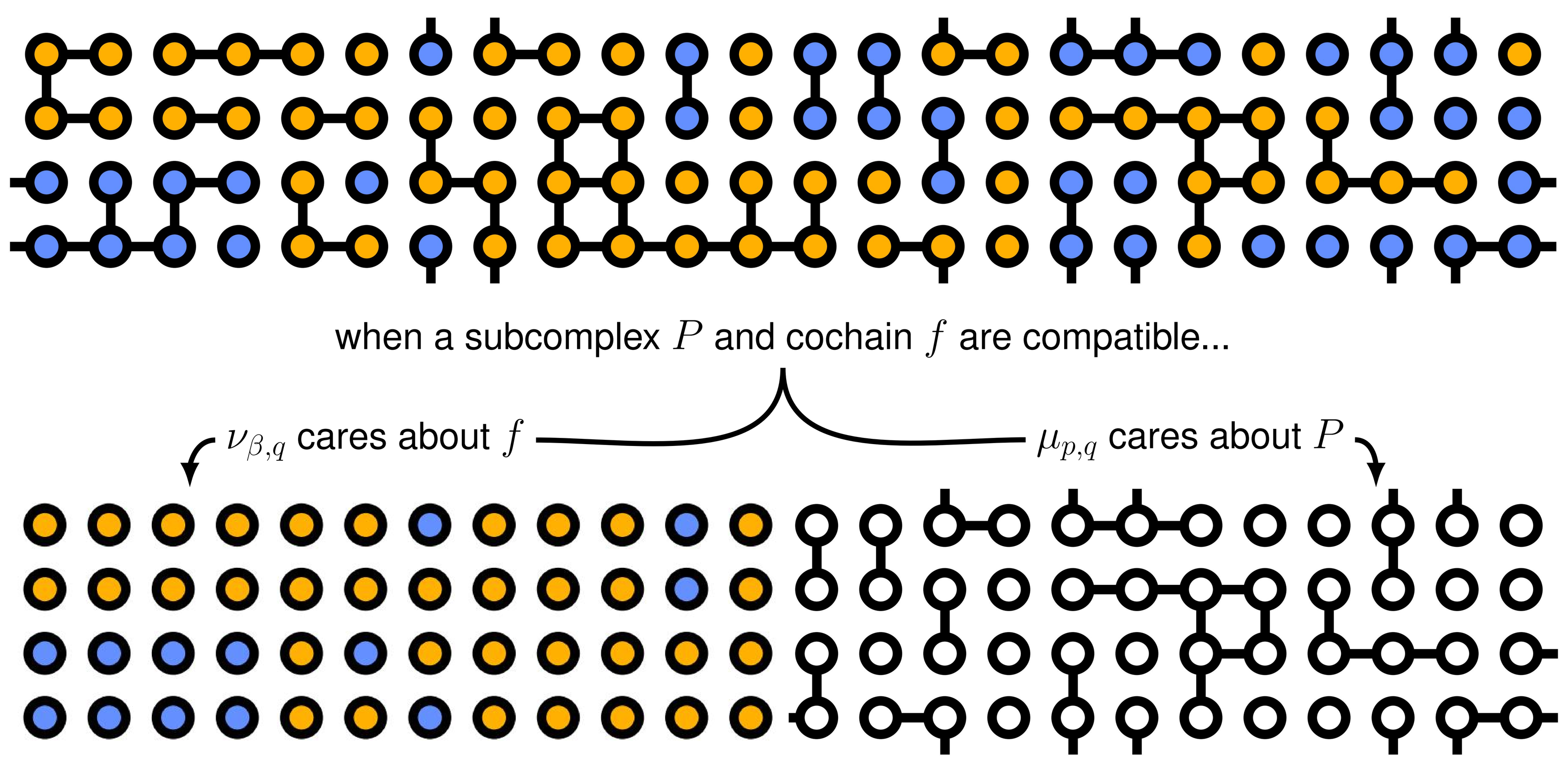

Figure 1. When $i=1$, X^i is a graph, so f is a vertex coloring and P is an edge-subgraph. Since $q=2$, PLGT is the ***Ising model***.

A cochain f and subcomplex P are ***compatible*** when f is a cocycle on P. The ***Edwards-Sokal coupling*** $\mathfrak{K}$ is a joint distribution over compatible pairs (f,P) [4–6].

> If (f,P) is a compatible pair, $\mathfrak{K}(-\mid P)$ ***is the uniform distribution over cocycles*** $Z^{i-1}(P;\mathbb{Z}/q\mathbb{Z})$, and $\mathfrak{K}(-\mid f)$ ***is independent percolation over*** f***-compatible*** i***-cells***.

Later, we use the coupling to build efficient sampling algorithms.

Detecting phase transitions with giant cycles

Giant i***-cycles are the nontrivial elements of*** $H_i(\mathbb{T}^d,\mathbb{Z}/q\mathbb{Z})$. ***Homological percolation is the event that*** $PH_i(P,\mathbb{Z}/q\mathbb{Z})$ ***is nontrivial*** [7, 8]. For $i=1$, the phase transition in PLGT occurs when giant 1-cycles appear in the PRCM (like in ***Figure 2***).

> When $d=2i$ and q is an odd prime, ***homological percolation in the PRCM happens at the self-dual point*** $p_{\mathrm{sd}}(q)=\sqrt{q}/(1+\sqrt{q})$ ***of PLGT***, which physicists believe is PLGT's critical point. We conjecture that ***PLGT's phase transition occurs exactly when homological percolation does in the PRCM*** [3].

We can use the coupling $\mathfrak{K}$, persistent homology, and linear algebra to generalize two algorithms for simulating these systems on ***scale-***N ***cubical*** d***-torus*** $\mathbb{T}^d_N$.

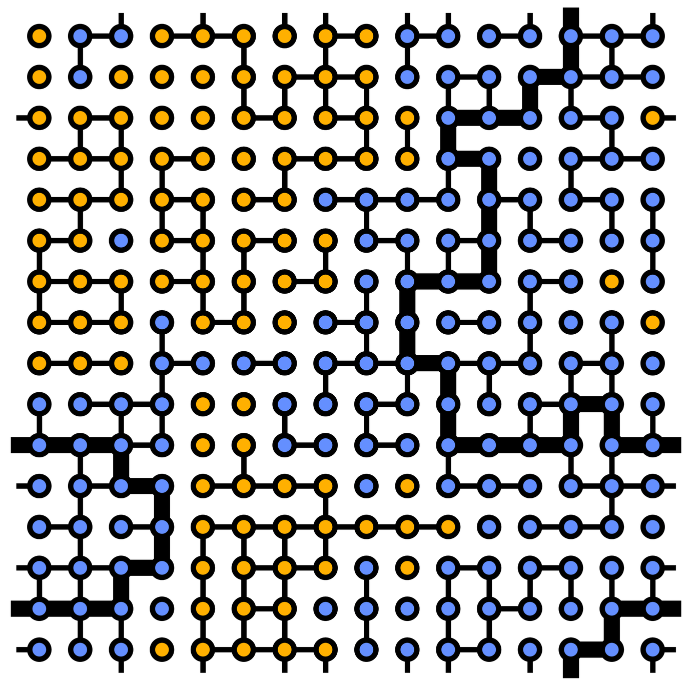

Figure 2.

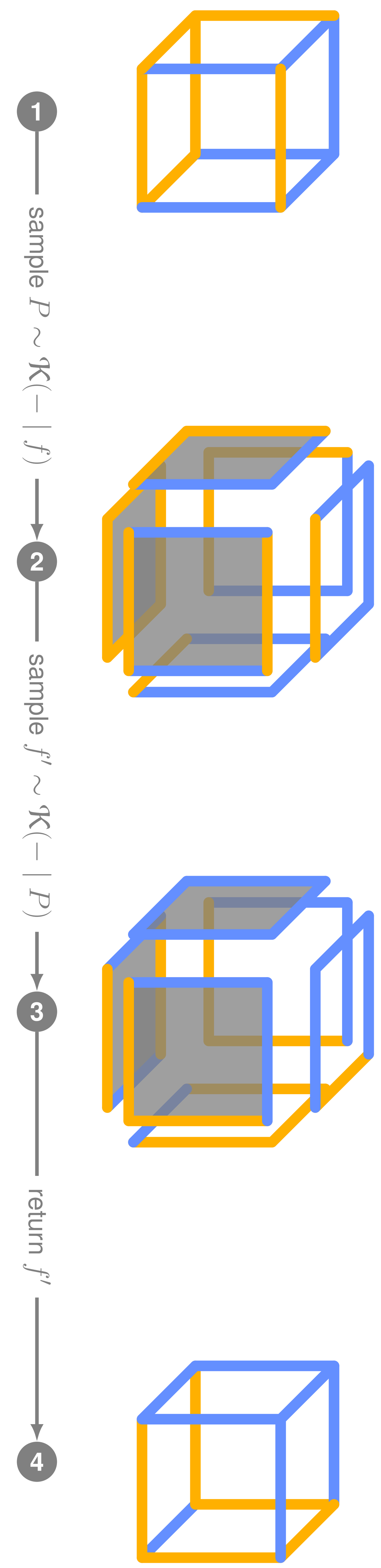

Figure 3. Both Algorithms run through the same ***core loop***, alternately sampling PRCM and PLGT.

Algorithms, performance, and future work

The ***plaquette invaded-cluster (PIC)*** [9, 10] and ***plaquette Swendsen–Wang (PSW)*** [10, 11] algorithms sample PLGT by iteratively updating a cochain f, each passing through a f-compatible subcomplex in a different way.

> Empirically, ***Algorithms 1 and 2 blow past standard algorithms for sampling PLGT and PRCM, even on extremely large systems*** [10]. See ***Figure 4***.

> ***PSW provably targets PLGT***. ***PIC estimates homological percolation thresholds for*** $\mathbb{T}^d_N$ [10].

We expect PIC to converge to PLGT as $N \longrightarrow \infty$, according to the scaling behavior in [3].

Algorithm 1 ***(PIC update routine).***

1. Let C be the set of f-compatible i-cells in X, and $P := X^{i-1}$. ❶
2. Pick uniform random i-cells in C and add them to P until $PH_i(P,\mathbb{Z}/q\mathbb{Z}) \neq \varnothing$. ❷
3. Sample and return $f' \sim \mathfrak{K}(-\mid P)$. ❸ ❹

Algorithm 2 ***(PSW update routine).***

1. Let C be the set of f-compatible i-cells in X, and $P := X^{i-1}$. ❶
2. Add each i-cell in C to P with probability $p = 1-e^{-\beta}$. ❷
3. Sample and return $f' \sim \mathfrak{K}(-\mid P)$. ❸ ❹

Subway markers (e.g. ❷) match the adjacent step to its stop on the core loop in ***Figure 3.***

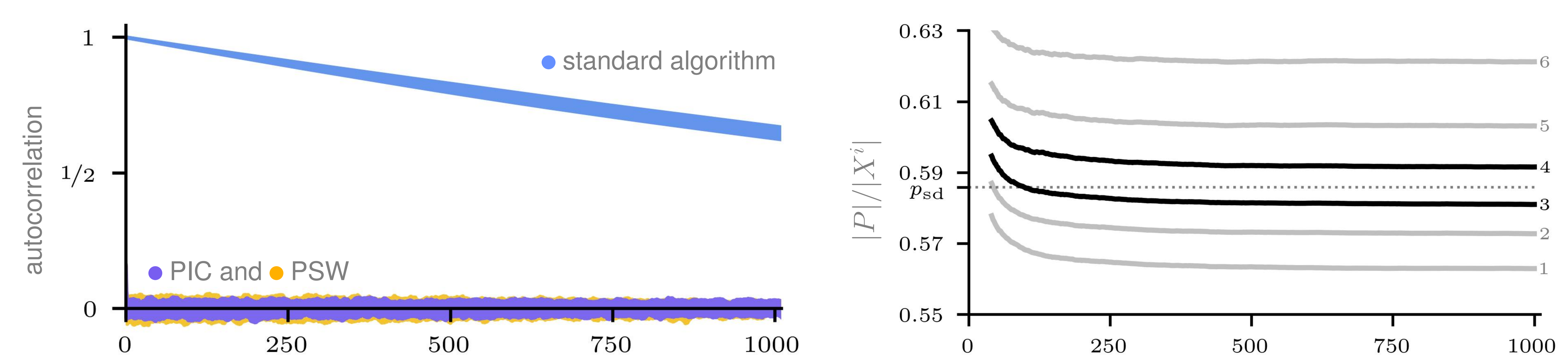

Figure 4. (Left) autocorrelation decay of $\mathbf{H}(f)$, comparing PIC and PSW to a standard algorithm called ***Glauber dynamics*** on $\mathbb{T}^4_{10}$ by measuring the similarity of (f_t,P_t) to the initial pair (f_0,P_0). (Right) expected thresholds for finding k independent cycles on $\mathbb{T}^4_{10}$.

> ***As the system size increases, we believe PIC will overtake PSW in convergence rate*** when the system is at or near criticality.

Both Algorithms are implemented in ATEAMS, our companion software [12]. Depending on q, we use PHAT or a modified twist-reduce for persistence, and LinBox for sparse linear algebra [12–14]. Currently, $\mathbb{T}^4_{40}$ is the largest system we have simulated successfully.

> ***Future work includes: running experiments on larger systems; implementing a probabilistic algorithm for persistence; further optimizing ATEAMS; estimating critical exponents; and extending to other scientific settings.***

References

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