Introduction. The Ising and Potts models for ferromagnetism are renowned for their comparative simplicity and theoretical depth. Using only a lattice graph, a weighting scheme, and a carefully concocted sampling mechanism, we can study important physical phenomena with remarkable fidelity to the real thing. Moreover, the Ising and Potts models are represented by special cases of the *random cluster model*, a broad class of random graph models [2]. These models and the use of *Markov chain Monte Carlo* (or *MCMC*) techniques to study them are longstanding areas of inquiry in physics, statistical mechanics, mathematics, and computer science. Recently, mathematicians and physicists have borrowed strategies from algebraic topology to widen the view of the random cluster model [1]. In addition to assisting with this work, I plan to update computational tools in kind: by generalizing contemporary MCMC algorithms, proving their correctness, and creating efficient programmatic implementations for them, we can enhance the already-rich data borne of the Potts and Ising models [4, 3]. These new tools let us investigate physical phenomena in a multi-dimensional setting, blazing new trails of discovery in statistical mechanics, physics, and mathematics.

Background. The Ising model is simple and prescriptive. Using finite lattice graphs L as prototypes for atomic structure, vertices and edges stand in for atoms and bonds. To mimic magnetism, each atom has a spin of "up" or "down," represented numerically as 0 or 1: our sample space is the set of all possible spin assignments $\sigma : V \to \{0, 1\}$, with V the vertex set of L. The chance of sampling any particular σ is proportional to $\exp(-\beta\, \mathbf{H}(\sigma))$ where $\mathbf{H}(\sigma)$ represents the *energy* of σ, the number of adjacent vertices with different spins; the importance of this energy is controlled by an *inverse temperature parameter* $\beta \geq 0$. The Potts model naturally extends the Ising model by allowing spins in $\{0, \ldots, q\}$, for q an integer.

Though procedures like Glauber dynamics simulate well-studied Markov chains and have favorable theoretical guarantees on these models, they are *slow-mixing*, as they require a large number of steps to converge to our desired probability measure. A fast-mixing alternative is the *Swendsen-Wang* algorithm [4]: given a finite lattice graph L, a probability $p = 1 - \exp(-\beta)$, and a spin configuration σ, we sample a random subgraph of L by first ignoring edges (u, v) when $\sigma(u) \neq \sigma(v)$ and including edges (x, y) with probability p when $\sigma(x) = \sigma(y)$. We then update σ by setting every vertex in each connected component to have the same, uniformly randomly chosen spin, and repeat this procedure for some number of steps. While this algorithm is destructive in that it "forgets" the starting spin configuration extremely quickly, it takes leaps and bounds across the state space, finding desirable spin configurations rapidly.

Intellectual Merit. My primary research goal is to fully generalize and implement a higher-dimensional analogue of the Swendsen-Wang algorithm, enabling the simulation of Markov chains on lattices built of higher-order cells. The generalization is grounded in robust topological theory: by treating our spin configurations σ as assignments on the k-dimensional faces that bound $(k+1)$-dimensional cells, we can closely study the k-dimensional substructures of our cubical lattice. In topological terms, these generalized configurations σ are equivalent to *cochains* on the k-dimensional cells of a finite cell complex X. Akin to the Potts model, the probability of sampling any particular σ is proportional to $\exp(-\beta\, \mathbf{H}(\sigma))$. Here, $\mathbf{H}(\sigma)$ counts the number of $(k+1)$-dimensional cells C of X where the sum of coefficients on C's k-dimensional faces is nonzero (mod q). This generalized probability measure is called the *k-dimensional Potts lattice gauge theory* on X [1].

Indeed, this construction aligns with our earlier definitions of the Ising and Potts models: the classical models arise when we set $k = 0$, as sampling a cocycle on the connected components of the graph is exactly the same as assigning the same spin to every vertex in each connected component [1]. In Figure 1, we can see the generalized Swendsen-Wang algorithm at work for $k = 1$. Describing these models in a dimension-agnostic way extends their scientific utility: we can now extract far richer information in an entirely general setting, amplifying our mathematical, physical, and statistical understanding of the complex systems they simulate.

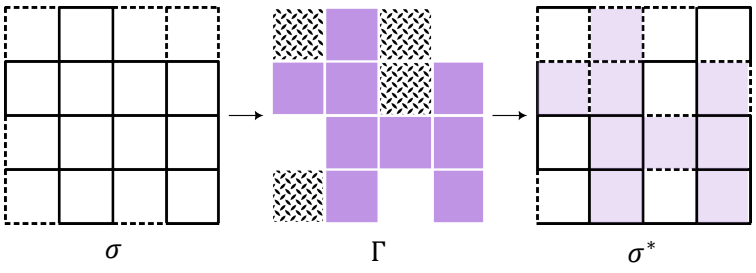

Figure 1: *a Swendsen-Wang evolution step sampling* 1*-dimensional Potts lattice gauge theory on a cubical complex. Edge coefficients are in* $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$*, drawn as solid or dashed lines. Starting with an edge configuration* σ*, we sample a subcomplex* Γ*: squares* γ *on which* $\sigma(\gamma) = 0$ *are included (purple) with probability* p *or excluded (hatched) with probability* $(1-p)$*; squares on which* $\sigma(\gamma) \neq 0$ *are ignored (blank). Once we construct* Γ*, we sample a new edge configuration* σ^* *which vanishes on each square of* Γ*, and repeat.*

In my work with Dr. Ben Schweinhart, I have already made meaningful strides in this area. Computationally, I have developed a highly customizable, mathematically-sound, open-source suite of software tools which allows users to efficiently simulate Markov chains on the Ising, Potts, and other random cluster variants using generalized sampling strategies. In the immediate future, I will build on this work by generalizing and implementing other simulation methods, like the *invaded-cluster* algorithm [3], incorporating them into our experimental repertoire. Additionally, I will markedly improve the high-performance numerical routines used in these sampling procedures, making large, complex experiments easily done on laptops instead of supercomputers. Mathematically, I plan to prove that these generalized procedures indeed target the distributions which characterize the random cluster model, alongside proving important properties like *irreducibility* (or *ergodicity*) and *reversibility* (or *detailed balance*), further cementing these models' foundational importance in statistical mechanics.

Broader Impacts. The random cluster model sits at the intersection of computer science, mathematics, and statistical mechanics, and advancements in one discipline enhance our understanding in all of them. Through theoretical development and lowering computational barriers to entry, we can democratize the knowledge we curate. From undergraduates to seasoned researchers, anyone with a computer can explore their ideas using open-source, scientifically sound software, conducting experiments on extremely — and, until recently, prohibitively — large systems. I am already making these goals concrete through my involvement with the Mason Experimental Geometry Lab (MEGL), George Mason's collaborative research incubator. As a graduate research mentor at MEGL, I help students understand complex mathematical concepts, learn what real mathematics research entails, and find their voices as young mathematicians. Currently, I mentor students on Dr. Michael Jarret's Quantum MCMC project, and plan to co-mentor a project at the intersection of percolation theory and topology with Dr. Schweinhart in coming semesters. I intend to continue in this role for the duration of my graduate study, deepening my investment in the mathematical community as a mentor, teacher, and researcher.

References.

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