Math 203 Matrix Algebra -- Exam II: Professor Gabel (April 23, 2002)

Directions: Do ALL of your work on these sheets of paper. No calculators are allowed. In order to maximize the credit you receive, show all of your steps, write neatly and give some reasons. There are two types of reasons: either by definition or by theorem. Make clear which you are using. Write reasons.

Remember, the honor code is to be observed on this exam. You are allowed one "crib" sheet on this exam.

[Suggestion: some of the problems say: show something. Be guided in your answer by how much space I have left for your work. In particular, nearly all of these problems require little space to solve. Think a little, think a bit more, and then start writing.] Problems marked NPC are No Partial Credit.

Except for the "MUST/MIGHT" questions, each part of each problem is worth 5 pts.
There are 100 points on this exam Make sure you have 4 pages and put your name on each page.

(1) NPC Each of the questions below is worth 2 points (making 20 points total). Circle T for True, F for False or circle blank if you wish to leave the problem blank.

SCORING: correct = +2, blank = +1 wrong = +0

T  F  blank Let B be a basis for \( \mathbb{R}^n \). As usual, if \( v \in \mathbb{R}^n \) denote by \([v]_B\) the coordinate vector of \( v \) with respect to the basis B. If \( x \) and \( y \) are in \( \mathbb{R}^n \) and if \([x]_B = [y]_B\), then \( x \) MUST equal \( y \).

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(2) [NPC] Let \( V = P_3 \), the vector space of all polynomials of degree 3 or less in the variable \( t \).

It is a fact that \( B = \{1, 1+t, 1+t^2, 1+t^3\} \) is a basis for \( V \). Find the coordinate vector of the vector \( t^2 \) with respect to this basis; that is, find \([t^2]_B\). [HINT: this vector has 4 components.]

\[
[t^2]_B = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \text{ since } t^2 = (-1)(1) + (0)(1+t) + (1) (1+t^2) + (0)(1+t^3).
\]
(3) Let \( A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \) [NOTE: I've corrected this by replacing 5 5 7 with 5 6 7 on the last row.]

(a) What is the rank of \( A \)? Give a reason. [No reason, no credit.]

\[ \text{rank } A = 1, \text{ as all the columns are multiples of the first one and so the first column is a basis for the span of the columns so that the dimension of the column space (which is the rank) equals 1. OR, there is one pivot.} \]

(b) What is the dimension of the null space of \( A \)? Give a reason. [No reason, no credit.]

\[ \text{dim (null } A \text{) = 6 because rank } A + \text{ dim (null } A \text{) = number of columns and rank } A = 6. \text{ OR, if you row reduce } A, \text{ there are 6 free variable, and so the dim of the null space =6} \]

NOTE: I GAVE CREDIT TO EVERYONE FOR THIS PROBLEM BECAUSE OF THE "ERROR" OF THE 5 5 7 VERSUS THE 5 6 7. IF YOU DID NOT GET FULL CREDIT FOR THIS ONE, SHOW ME YOUR EXAM.

(4) Assume \( A \) is a matrix that is row equivalent to the matrix \( B = \begin{pmatrix} 1 & 0 & 2 & 3 & 1 \\ 0 & 1 & -1 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \).

Complete the following sentences, or write NMI [Need More Information.]

(a) A basis for the column space of \( A \) equals:

NMI OR the first and second columns of \( A \).

(b) A basis for the column space of \( B \) equals:

The first two columns of \( B \) [because they are the pivot columns]. Actually, any two column of this matrix are a basis for the column space of \( B \) because, since there are two pivots, the dim of the col space is 2 and, as it turns out, any two of the columns are independent.

(c) The dimension of the null space of \( A \) equals:

Since \( A \) is row equivalent to \( B \), \( A \) and \( B \) have the same null space. Since \( B \) has 3 free variable, \( \text{dim (null } B \text{)} = 3 \), and so \( \text{dim (null } A \text{)} = 3 \)

(d) The dimension of the null space of \( B \) equals:

\[ \text{dim (null } B \text{) = 3} \text{ [see part (c).]} \]
(5) [NPC] Let A be a 3x3 matrix with eigenvalues 5 and 7.

Assume \( \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix} \) is a basis for the eigenspace for the eigenvalue 5 and \( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \) is a basis for the eigenspace of 7.

We know, then, (and you may assume) that \( A = PDP^{-1} \).

Then \( P = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix} \) and \( D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} \).

(6) Let \( A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix} \).}

(a) [NPC] What is the characteristic polynomial of \( A \)?

\( (3-\lambda)^2(5-\lambda)(2-\lambda) \)

(b) [NPC] What is the multiplicity of 5?

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(c) Find the dimension for the eigenspace corresponding to the eigenvalue 3. Show all of your work. Give a reason for an answer. A number, alone, correct or not, will receive no credit.

The eigenspace of 3 is the null space of \( A-3I \). If you compute this matrix, you'll see that there are pivots in columns 3,4,5, 6 and thus there are two free variables. Thus, the null space has dimension 2

(d) Find a basis for the eigenspace corresponding to the eigenvalue 2. Show, explain, your reasons.

We need find a basis for the solutions to \((A-2I)(x)=0\), for this is the eigenspace of 2. If you compute \( A-2I \), you'll see that there is one free variable, namely, \( x_6 \).

So, the solution is of the form \( \begin{pmatrix} x_6 \\ \vdots \end{pmatrix} \).

Following our standard method for solving homogeneous systems, we get the solution to be:

\( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \), and so \( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \) is a basis for the eigenspace of 2.
(7) [NPC] What are the eigenvalues of the matrix \( A = \begin{pmatrix} 4 & -1 \\ 3 & 0 \end{pmatrix} \). [Watch your minus signs! Show all your work.]

\[
det (A-\lambda I) = \det \begin{pmatrix} 4-\lambda & -1 \\ 3 & -\lambda \end{pmatrix} = (4-\lambda)(-\lambda)-(-1)(3) = (\lambda-1)(\lambda-3), \]
which has roots 1 and 3, and so 1 and 3 are the eigenvalues.

(8) Let \( A \) be square matrix and let \( v \) and \( w \) be non-zero vectors such that \( Av = v \) and \( Aw = 3w \). Finish the argument that follows, and thus prove that \( v+w \neq 0 \). [DON'T use a theorem about distinct eigenvectors.]

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Proof:

In order to show \( v+w \neq 0 \), we will assume \( v+w=0 \) and reach a contradiction.

So, assume \( v+w=0 \). Then \( A(v+w) = A(0) \). Thus: ….

[now - you finish the argument by using \( A(v+w) = A(0) \), \( Av = v \) and \( Aw = 3w \) to get a contraction. Use words and sentences!!]

So, \( Av+Aw = 0 \), as \( A(0) = 0 \).

But we are given that \( Av=v \) and \( Aw=3w \), so we get \( v+3w=0 \).

Now, since we've assumed that \( v+w=0 \), we have \( v=-w \), and so \( v+3w \) becomes \( -w+3w=0 \).

Thus we have \( 2w=0 \), and so \( w=0 \). But, our hypothesis is that \( v \) and \( w \) are non-zero, and we have just shown \( w=0 \). This is a contradiction, and so our assumption that \( v+w=0 \) cannot be correct. Thus, \( v+w \neq 0 \), and we are done.

(9) Let \( V \) be a vector space of dimension 2. Assume \( \{u,v\} \) is a basis of \( V \). Let \( H = \text{Span} \{u, u+v\} \).

Find \( \text{dim} \ H \). [Show your work. Explain your reasoning.]

We claim that \( \text{dim} \ H = 2 \). Since \( H \) is a subspace of \( V \), its dimension must be \( \leq 2 \), so need only find 2 independent vectors in \( H \). Well, we claim that \( u \) and \( u+v \) (which are clearly in \( H \)) are independent. To show this we must show that \( a(u) + b(u+v) = 0 \) forces \( a=b=0 \). Well, if \( a(u) + b(u+v) = 0 \), then \( (a+b)u + bv = 0 \). But, \( u \) and \( v \) are independent as they are given to be a basis of \( V \). Thus \( a+b=0 \) and \( b=0 \). But, then clearly, \( a=b=0 \). So, \( u \) and \( u+v \) are independent and thus must form a basis of \( H \). So \( \text{dim} \ H = 2 \).