Instructions: Present well-organized and neat solutions. (Please present your solutions in order. For example, your solution to Problem 2 should precede your solution to Problem 3, and if a problem has two parts, part (a) should come before part (b). As always, use paper which is approximately 8.5 inches by 11 inches, present tidy and easy-to-follow solutions, draw boxes around or highlight your final answers (but don’t just give answers without supporting work), and staple all sheets together in the upper left hand corner. I don’t like cover sheets, executive summaries, folders, binders, or paper clips.)

Note: If I request a numerical estimate, give either the exact value, or a value which has been rounded to three significant digits. (23.6, 0.236, and 0.00236 all have three significant digits. 0.024 only has two significant digits.)

Note: For maximum likelihood estimates/estimators, be sure to clearly establish that you’ve obtained a maximizing value. (E.g., if you set the derivative of the likelihood function equal to 0, be sure to establish that the solution is a maximizing value and not a minimum or a point of inflection, and additionally argue that you’ve found the global maximum and not just a local maximum.) Also, if I request an estimate or estimator which doesn’t exist, you should fully explain why the estimate or estimator cannot be given.

Note: Below I use MLE to denote maximum likelihood estimator and mle to denote maximum likelihood estimate. Similarly, I use MME to denote method of moments estimator and mme to denote method of moments estimate. Estimators should be expressed using upper-case (e.g., \( X_i \)), and estimates should be expressed using upper-case (e.g., \( x_i \)). When writing likelihood functions, lower-case should be used (e.g., \( x \)). (Be sure to write clear enough so that \( X_i \) can be distinguished from \( x_i \).)

1) Let \( X_1, X_2, \ldots, X_n \) be iid random variables having pmf
\[
f_X(x|\theta) = \theta^{x-1}(1 - \theta) I_{\{1,2,3,\ldots\}}(x),
\]
where \( \theta \in \Theta = (0, 1) \).
(a) (5 points) Give a MME of \( \theta \) based on the first sample moment. (Note: The discrete distribution is that of the number of iid Bernoulli (\( \theta \)) trials needed to observe the first failure, and so it’s similar to the usual geometric distribution (for the number of trials needed to observe the first success). Perhaps this similarity will suggest a way to obtain the distribution mean.)
(b) (5 points) For the case of \( \sum_{i=1}^{n} x_i > n \), give the mle of \( \theta \). (Note: For the case of \( \sum_{i=1}^{n} x_i = n \), the mle of \( \theta \) doesn’t exist.)

2) \( X_1, X_2, \ldots, X_n \) are iid random variables having pdf
\[
f(x|\theta) = \frac{1}{\theta} \exp \left( 1 - \frac{x}{\theta} \right) I_{[\theta, \infty)}(x),
\]
where \( \theta > 0 \).
(a) (6 points) Give the MLE of \( \theta \).
(b) (4 points) Give the expected value of the MLE of \( \theta \) (the MLE of part (a)). (Comment: As a check of your work, you should find that the MLE is biased, but that the bias goes to 0 as \( n \) tends to infinity.)
(c) (2 points) Give the MLE of \( 5\theta^2 \) (which is the 2nd moment of the \( X_i \)).

3) \( X_1, X_2, \ldots, X_n \) are iid random variables having pdf
\[
f(x|\alpha, \beta) = \frac{\beta \alpha^\beta x^{\beta-1}}{I(\alpha + x)^{\beta+1}} I_{(0, \infty)}(x),
\]
where \( \alpha > 0 \) and \( \beta > 2 \).
(a) (6 points) Give mmes for \( \alpha \) and \( \beta \) based on the first and second sample moments, and evaluate these for the \( x_1 = 0.7 \), \( x_2 = 4.0 \), and \( x_3 = 0.1 \).
(b) (1 point) Why does one need to assume that \( \beta \) exceeds 2 in order to sensibly apply the method of moments as prescribed in part (a) (i.e., using the 1st and 2nd sample moments)?
4) $X_1, X_2, \ldots, X_n$ are iid random variables having pdf

$$f(x|\theta) = \frac{(1 + \theta x)}{2} I_{[-1,1]}(x),$$

where $\theta \in \Theta = [-1, 1]$.

(a) (4 points) Give the MME of $\theta$ based on the first sample moment.

(b) (6 points) Now consider the case of $n = 1$, and give a formula for the mle of $\theta$ which results from a single observed value, $x$. (You should find that the mle isn’t unique if $x = 0$ — more than one value of $\theta \in \Theta$ maximizes the likelihood.)

(c) (6 points) Now consider the case of $n = 4$, and particular sample of values $x_1 = 0.5$, $x_2 = -0.1$, $x_3 = 0.9$, and $x_4 = -0.5$, and give a numerical value for the mle of $\theta$, rounding to the nearest thousandth. (I suggest that you work with the likelihood, as opposed to the log-likelihood, for this problem. If you plug in the values for the $x_i$ and multiply the factors, you’ll get a fourth degree polynomial for the likelihood. To maximize this, you can set its derivative to 0 and obtain a root of the equation utilizing Newton’s method, using the mme (plug into your part (a) result) as an initial value. If you do this (correctly), you’ll converge to the mle in just a few iterations. (It would be nice if you also plotted the likelihood function in order to see that your solution is indeed the maximizing value of $\theta \in \Theta$.) Alternatively, you may choose to use some sort of software to obtain the mle (but if you do that, be sure to provide me with some sort of justification for your final answer).

5) Consider independent trials, with

- a type 1 outcome occurring with probability $p_1 = \theta^2$,
- a type 2 outcome occurring with probability $p_2 = \theta^2$,
- a type 3 outcome occurring with probability $p_3 = (1 - 2\theta)^2$,
- a type 4 outcome occurring with probability $p_4 = 2\theta^2$,
- a type 5 outcome occurring with probability $p_5 = 2\theta - 4\theta^2$,
- and a type 6 outcome occurring with probability $p_6 = 2\theta - 4\theta^2$,

where $\theta \in \Theta = (0, 1/2)$. Consider a sample of size 6 for which the various types occur with these counts: $n_1 = 1$, $n_2 = 1$, $n_3 = 0$, $n_4 = 2$, $n_5 = 1$, and $n_6 = 1$.

(a) (6 points) Based on this sample, give an estimate of $\theta$ using the method of maximum likelihood.

(b) (4 points) Based on the sample, give an estimate of $\theta$ using a frequency substitution estimator which uses $p_4$, but not the other sample proportions.