CONTACT LENS AND TEAR FILM DYNAMICS DURING BLINKING

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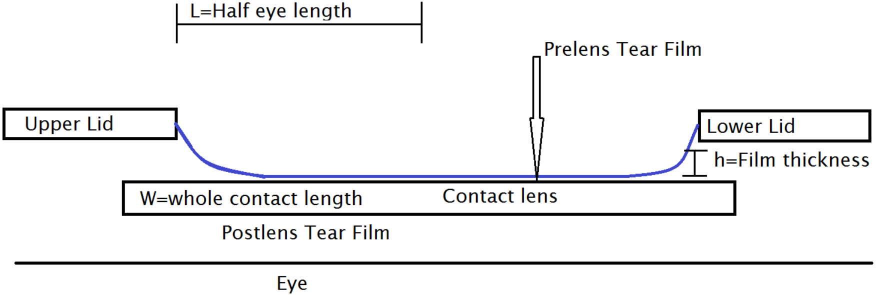
November 21st, 2016

Introduction

- The goal of the research is to calculate a numerical solution to a system partial differential equations that model tear film thickness on a contact lens in a blinking eye
- Our motivation is to understand the influence of contact lens motion on the tear film
- The computational method involves solving a nonlinear PDE coupled to an ODE for contact lens motion, and the equations are on a domain that changes size

Overview of the problem



- The partial differential equations model the thickness of the prelens film
- Blinking causes the contact lens to move
- Contact lens motion and blinking both influence tear film thickness and the flux boundary conditions

Tear film thickness model

- Model is based on Navier-Stokes equations and lubrication theory assumptions
- It was originally derived by [Corsaro, 2014; Horton, 2014; Anderson, June 2016]
- The equation for tear film thickness h is

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[\left(\frac{h^3}{3} + \beta_{CL} h^2\right)\left(\frac{\partial p}{\partial x} - G\right) - hU_{CL}\right]$$

- Where pressure p is

$$p = -\frac{1}{Ca}\frac{\partial^2 h}{\partial x^2}$$

- With parameters:

G	Gravity	β_{CL} and β_{Lid}	Slip Coefficients
Ca	Capillary number $\left(\frac{1-x}{1-X(t)}\frac{dX(t)}{dt}\right)$	U_{CL}	Contact lens speed (ODE)

Eyelid motion

- Eyelid motion can be modeled with a sinusoidal function or a piecewise parabolic function [Heryudono et al., 2007; Braun and King-Smith, 2007]
- The sinusoidal version is

$$X(t) = -\lambda - (1 - \lambda)\cos(t)$$

- λ is the percentage that the eye stays open
- $\lambda = .1$ is a mostly complete blink (only 10% stays open)
- $\lambda = .5$ is a half blink
- There are other models for eyelid motion that are more realistic but are more computationally challenging

Contact lens motion

- Originally derived by [Anderson, June 2016; Corsaro, 2014; Horton, 2014]
- The forces on the contact lens are from Couette flow assumptions

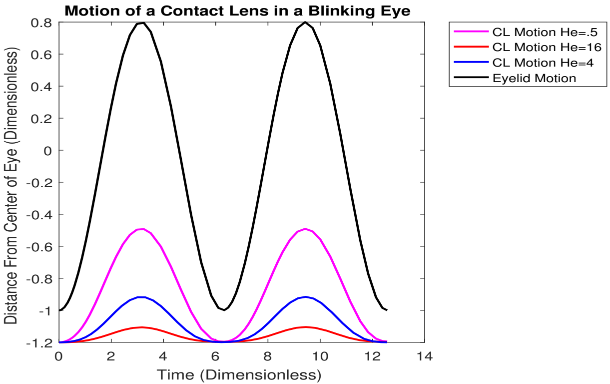
$$M\frac{dU_{CL}}{dt} = F_{Upper} + F_{Lower} + F_{Prelens} + F_{Postlens}$$

- The ODE can be simplified when $F_{Prelens} = 0$

$$U_{CL} = \frac{dC(t)}{dt} = -\frac{WP_1}{D+\beta_{CL}+\beta_{Lid}}\left(C(t)-C(0)\right) + \frac{P_1}{2\left(h_e+\beta_{CL}+\beta_{Lid}\right)}\left[\left(X(t)-C(t)\right)^2-\left(X(0)-C(0)\right)^2\right] - \frac{P_1}{2\left(h_e+\beta_{CL}+\beta_{Lid}\right)}\left[\left(C(t)+W+1\right)^2-\left(C(0)+W+1\right)^2\right].$$

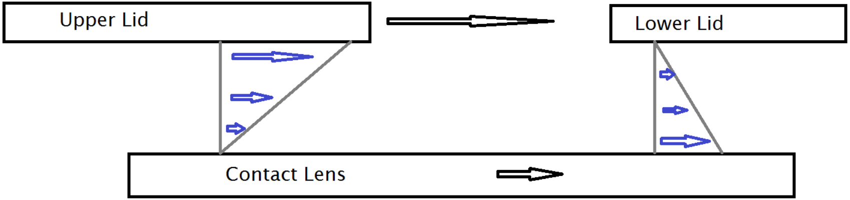
Examples of contact lens motion

This image shows the relation between eyelid motion and contact lens motion for varying values of h_e, the distance between the eyelid and the contact lens



Flux boundary conditions

- Flux boundary conditions are imposed at the eyelids on the first derivative of pressure, p
- Couette flow of the tear film caused by the motion of the eyelid and the contact lens causes tear film fluid to flow in and out of the domain during blinking



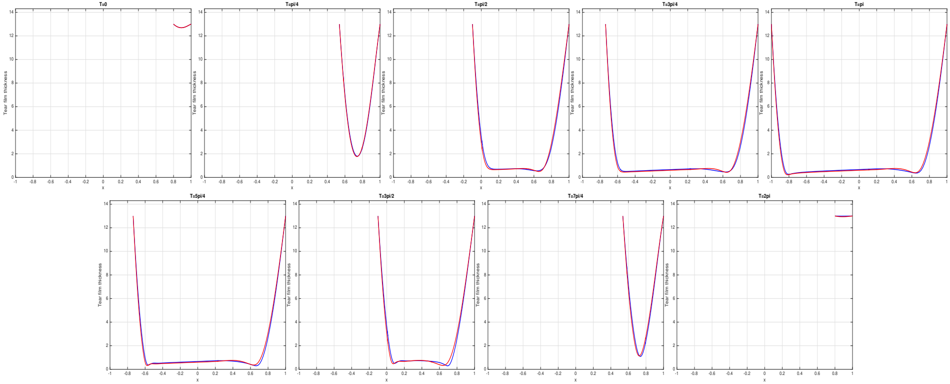
Numerical method

- The spatial derivatives are approximated using a modified Chebyshev spectral method [Heryudono et al., 2007]
- The Chebychev interpolation points are mapped to reduce interpolation error and speed up computation time [Kosloff and Tal-Ezer, 1993]
- The spectral differentiation discretizes the PDE into a system of differential algebraic equations (DAE), and the contact lens ODE is included in the DAE system
- The time derivatives are solved using a standard ODE solver on the DAE system
- A change of variables is used to deal with the moving domain [Heryudono et al., 2007]

$$X(t) \leq x \leq 1 \quad \Longrightarrow \quad \hat{x} = 1 - 2\frac{1-x}{1-X(t)} \quad \Longrightarrow \quad -1 \leq \hat{x} \leq 1$$

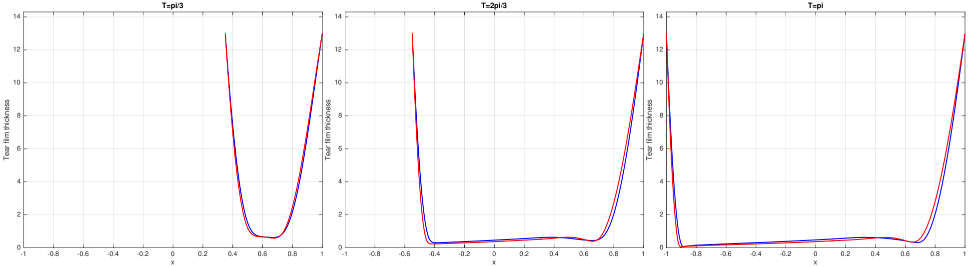
Sinusoidal blinking

A comparison of tear film thickness with (blue) and without (red) a contact lens



Effect of contact lens

- These plots compare tear film thickness with (blue line) and without (red line) a contact lens and with no flux
- The results without a contact lens are a replication of the results in [Braun and King-Smith, 2007]
- The contact lens drags the tear film in the direction that it is moving



Effect of contact lens (Zoomed In)

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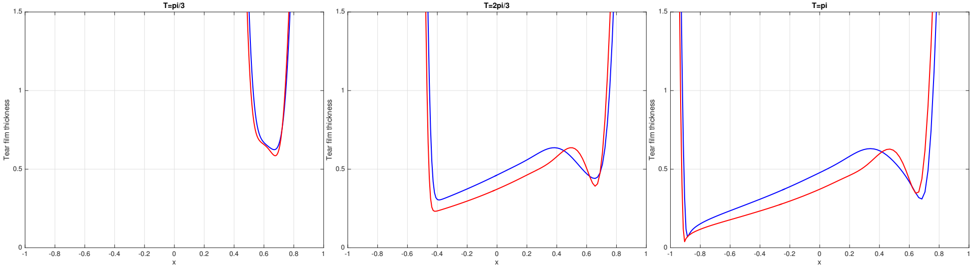


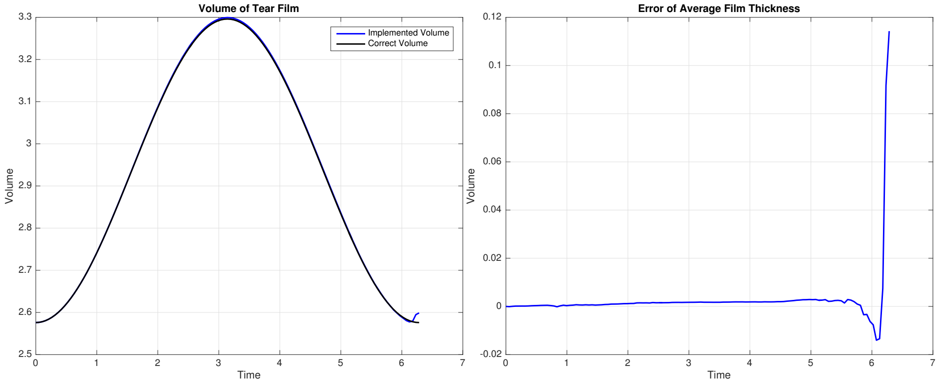
Sinusoidal blinking animation

This is tear film thickness in two sinusoidal blinks with a contact lens (blue line) and without (red line)

Error Analysis

- The flux boundary conditions allow us to know the volume of the tear film
- A numerical integral can be calculated for the tear film to find its implemented volume
- The implemented and expected volume can be compared to compute the error in volume
- The volume error can be divided by the length of the domain to find the average error in tear film thickness at any time

Error Analysis Graphs



Conclusions and Future work

- The contact lens moves in the same direction as the upper eyelid but to a smaller magnitude
- The main effect of a contact lens is to move the the center of mass of the tear film in the direction contact lens motion
- In the future we will use a more realistic eyelid motion model
- We will compare the effect of contact lens motion to an up/down saccade

Acknowledgements

- This research was funded by the NSF/Extreems program (Grant DMS-1407087)
- My travel was funded by the American Physical Society

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Parameters used in contact lens motion ODE

Parameter	Equal to	Meaning	Sample Value
C	$C(t) \times L$	Contact lens position	See ODE
C_0	$C(t)_0 \times L$	Contact lens initial position	$-.006$ Meters
W	$W \times L$	Contact lens length	.012 Meters
D	$D \times d$	Thickness of postlens film	2 Microns
h_e	$h_e \times d$	Distance between contact and eyelid	4 Microns
X	$X(t) \times L$	Upper eyelid position	Time dependent
X_0	$X(t)_0 \times L$	Upper eyelid initial position	$-.005$ Meters
P_1	$\mu L^2 / (mU_m d)$	Dimensionless parameter	27.008
L		Half eye length	.005 Meters
d		Characteristic thickness	5 Microns
μ		Tear viscosity	.0013 Meters
U_m		Mean blink speed	.2 Meters/sec
m	$\rho_w \times H \times W$	Contact lens mass	.0012 Kilograms
ρ_w		Density of water	1000 kg/m^3
H		Contact thickness	100 Microns

Modified Chebyshev Spectral Method

- Problems that have solutions that are smooth throughout the domain can have their Chebyshev points mapped to a new set of points [Kosloff and Tal-Ezer, 1993] [Heryudono et al., 2007]
- A symmetrical version of the mapping function is

$$x_{new} = \frac{\sin^{-1}(\alpha x_{cheb})}{\sin^{-1}(\alpha)}$$

  where α is a parameter that determines how much the points are spread
- The new set of points have less points clustered at the edges than the regular Chebyshev points
- This increases the stability of the numerical method, speeding up its solution and decreases the error of high order derivatives