Nonlinear Input-Output Equilibrium

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Abstract

We introduce and study Nonlinear Input-Output Equilibrium (NIOE). The main difference between NIOE and the classical Wassily Leontief's input-output (IO) model is: both the production cost and the consumption are not fixed and are not given a priori.

Instead, the production cost is an operator, which maps the output into the cost per unit and the consumption is an operator, which maps the price for goods into the consumption.

The NIOE finds such output and such prices for goods that the following two important goals can be reached.

First, at the NIOE the production cost is in agreement with the output and the consumption is in agreement with the prices for goods.

Second, at the NIOE the total production cost reaches its minimum, while the total consumption reaches its maximum.

Finding NIOE is equivalent to solving a variational inequality (VI) on a simple feasible set $\Omega = \mathbb{R}^n_+ \times \mathbb{R}^n_+$.

Therefore for finding NIOE we use two first-order methods: Pseudo-Gradient Projection (PGP) and Extra Pseudo-Gradient (EPG), for which the projection on Ω is the main operation per step. Numerically both methods require two matrix by vector multiplications or $O(n^2)$ arithmetic operations per step.

It allows, along with proving convergence and global Q-linear convergence rate, also establishing complexity bounds for both methods under various assumptions on the input data.

1. Introduction

The input-output (IO) model has been introduced before the second World War (WW2). The main purpose was better understanding the interdependence of the production sectors of an economy.

Since WW2 the IO model has been widely used for analysis of economic activities, planning production, economic prognoses, international trade just to mention a few. The applications of IO range from a branch of a single economy to the World economy.

The main contributor to IO theory and, in particular, to the practical aspects of IO model was Wassily W. Leontief (1906-1999)(see [17]-[19]).

He received the Nobel Price in Economics in 1973 "for the development of the input-output method and for its application to important economic problems."

The input-output model assumes that an economy produces n products, which are also consumed in the production process. The elements $a_{ij} \geq 0$ $1 \leq i \leq n$; $1 \leq j \leq n$ of the consumption matrix $A = \|a_{ij}\|$ show the amount of product $1 \leq i \leq n$ consumed for production of one unit of product $1 \leq j \leq n$.

Let $x = (x_1, ..., x_n)$ be the production vector, i.e. x_j defines how many units of product $1 \leq j \leq n$ we are planning to produce. Then Ax is the total consumption needed to produce the vector $x \in \mathbb{R}^n_{++}$. Therefore the components of the vector $y = x - Ax = (I - A)x$ shows how much of each product is left after the production needs were covered. Vector y can be used for consumption, investments, trades just to mention a few possibilities.

Let $\lambda = (\lambda_1, ..., \lambda_n)$ be the price vector, i.e. λ_j defines the price of one unit of product $1 \leq j \leq n$, then $q = (I - A)\lambda$ is the profit vector, i.e. q_j defines the profit out of one unit of product $1 \leq j \leq n$ under the price vector $\lambda \in \mathbb{R}^n_{++}$.

The IO model solves two basic problems.

(1) For a fixed and given a priori consumption vector $c = (c_1, ..., c_n)^T \in \mathbb{R}^n_+$ the IO finds the production vector $x_c = (x_{c,1}, ..., x_{c,n})^T \in \mathbb{R}^n_+$, which guarantees the required consumption. Vector x_c is the solution of the following linear system

$$(I - A)x = c, \ x \in \mathbb{R}^n_+ \tag{1.1}$$

(2) For a fixed and given a priori profit vector $q = (q_1, ..., q_n)^T \in \mathbb{R}^n_+$ the IO finds such price vector $\lambda_q = (\lambda_{q,1}, .., \lambda_{q,n})^T \in \mathbb{R}^n_+$, which guarantees the given profit. Vector λ_q is the solution of the following system

$$q = (I - A)^T\lambda, \ \lambda \in \mathbb{R}^n_+. \tag{1.2}$$

For a **productive** economy both systems (1.1) and (1.2) have unique solutions and both vectors x_c and λ_q are positive (see, for example, [2], [8]). The productivity assumption will be kept throughout the paper.

Another possible use of IO model consists of finding "optimal" production vector $x^* \in \mathbb{R}^n_+$ and "optimal" price vector $\lambda^* \in \mathbb{R}^n_+$ by solving the dual Linear Programming (LP) pair.

Let $p = (p_1, ..., p_n)^T$ be fixed and a priori given production cost vector, and $c = (c_1, ..., c_n)^T \in \mathbb{R}^n_+$ be fixed and a priori given consumption vector. Finding "optimal" production output x^* and "optimal" consumption prices λ^* leads to the following dual pair of LP problems

$$x^* \in \text{Argmin}\{(p, x) | (I - A)x \geq c, x \in \mathbb{R}^n\} \tag{1.3}$$

and

$$\lambda^* \in \text{Argmax}\{(c, \lambda) | (I - A)^T\lambda \leq p, \lambda \in \mathbb{R}^n_+\} \tag{1.4}$$

For a productive economy both the primal feasible solution

$$x_c = (I - A)^{-1}c$$

and the dual feasible solution

$$\lambda_p = ((I - A)^T)^{-1}p$$

are positive vectors (see, for example, [2], [16]).

So x_c and λ_p are primal and dual feasible solution, for which the complementarity conditions are satisfied. Therefore $x^* = x_c$ and $\lambda^* = \lambda_p$.

In other words, the dual LP pair (1.3) and (1.4), unfortunately, does not produce results different from IO model.

The purpose of this paper is to introduce an study the Nonlinear Input-Output Equilibrium (NIOE), which extends the abilities of the classical IO model in a few directions.

The fixed cost vector $p = (p_1, ..., p_n)^T \in \mathbb{R}^n_+$ we replace by a cost operator $p : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, which maps the production output vector $x = (x_1, ..., x_n)^T \in \mathbb{R}^n_+$ into cost per unit vector $p(x) = (p_1(x), ..., p_n(x))^T \in \mathbb{R}^n_+$.

Similarly, the fixed consumption vector $c = (c_1, ..., c_n)^T \in \mathbb{R}^n_+$ we replace by consumption operator $c : \mathbb{R}^n_+ \to \mathbb{R}^n_+$,which maps the prices for goods vector $\lambda = (\lambda_1, ..., \lambda_n)^T \in \mathbb{R}^n_+$ into consumption vector $c(\lambda) = (c_1(\lambda), ..., c_n(\lambda))^T \in \mathbb{R}^n_+$.

We will call a pair $y^* = (x^*; \lambda^*) \in \Omega = \mathbb{R}^n_+ \times \mathbb{R}^n_+$ the Nonlinear Input-Output Equilibrium (NIOE) if

$$x^* \in \text{Argmin}\{(p(x^*), x) | (I - A)x \geq c(\lambda^*), x \in \mathbb{R}^n_+\} \tag{1.5}$$

and

$$\lambda^* \in \text{Argmax}\{(c(\lambda^*), \lambda) | (I - A)^T \lambda \leq p(x^*), \lambda \in \mathbb{R}^n_+\} \tag{1.6}$$

The primal and dual LP (1.3) and (1.4), which one obtains in case of $p(x) \equiv p$ and $c(\lambda) \equiv c$, can be viewed as linear input-output equilibrium, which is identical to IO model.

Finding NIOE $y^* = (x^*; \lambda^*)$ is equivalent to solving a concave two person game, which is, in turn, equivalent to solving a particular variational inequality (VI) on $\Omega = \mathbb{R}^n_+ \times \mathbb{R}^n_+$.

Using the equivalence we found conditions on the production and consumption operators, under which the NIOE exists and it is unique.

For solving VI we use two projection type methods. Projection methods for convex optimization were introduced in the 60s (see [11], [20]) and used for solving VI in [4]. They got mainly theoretical value, because even in case of linear constraints projection methods require solving quadratic programming problem at each step. Projection on Ω, however, is a very simple operation, therefore for finding $y^* = (x^*; \lambda^*) \in \Omega$ we consider two first order projection type methods: Pseudo-Gradient Projection (PGP) and Extra Pseudo-Gradient (EPG), for which projection on Ω is the main operation per step. Numerically it requires two matrix by vector multiplications, which is at most $O(n^2)$ arithmetic operations per step.

It allows, along with proving convergence and establishing global Q-linear convergence rate, also estimate complexity bounds for both methods under various assumptions on the input date.

Both methods decompose the NIOE problem in the primal and in the dual spaces allowing computing the production and the price vectors simultaneously. Both PGP and EPG can be viewed as pricing mechanisms for establishing NIOE.

The main distinction of the EPG is its ability to handle NIOE problems when both the production and the consumption operators are not strongly monotone, but just monotone.

This is one of the finest property of the EPG method introduced by G.Korpelevich in the 70s (see [15]). It drawn much attention lately (see, for example, [1], [5]-[7],[12]-[14], [21]).

In case of NIOE the application of EPG leads to two stage algorithm. At the first stage EPG predicts the production vector $\hat{x} \in \mathbb{R}^n_+$ and the price vector $\hat{\lambda} \in \mathbb{R}^n_+$. At the second stage EPG corrects them dependent on the production cost per unit vector $p(\hat{x})$ and consumption vector $c(\hat{\lambda})$.

The paper is organized as follows. In the following section we recall some basic facts about IO model.

In section 3 we show the equivalence of NIOE to two person concave game and, eventually, to a particular VI and prove existence and uniqueness of NIOE.

In section 4 we consider the PGP method and show its global Q-linear convergence rate under local strong monotonicity and local Lipschitz condition of both p and c operators.

In section 5 we prove convergence of the EPG for finding NIOE under minimum assumption on the input data.

In section 6 under local strong monotnicity and Lipschitz condition of both operators p and c the global Q-linear convergence rate was proven and complexity bound for EPG method was established. In section 7 we estimate the Lipschitz constant of the VI operator.

We conclude the paper with some remarks on complexity and numerical aspects of PGP and EPG methods.

2. Preliminaries

We consider an economy, which produces n products. The products are partially consumed in the production process.

The economy is called **productive** if for any consumption vector $c \in \mathbb{R}^n_+$ the following system

$$x - Ax = c, \; x \geq 0 \tag{2.1}$$

has a solution.
In such case the matrix A is called productive matrix. It is rather remarkable that if the system (2.1) has a solution for only one vector $c \in \mathbb{R}^n_{++}$ it has a solution for any given vector $c \in \mathbb{R}^n_+$.

It is obvious that for any consumption vector $c \in \mathbb{R}^n_+$ the system (2.1) has a positive solution if the inverse matrix $(I - A)^{-1}$ exists and it is positive.

To address the issue of matrix $B = (I - A)^{-1}$ positivity we have to recall the notion of indecomposability of a non-negative matrix A.

Let S be a subset of a set of indices $N = \{1, ..., n\}$, i.e. $S \subset N$ and $S^{'} = N \setminus S$. The sets of indices S is called isolated set if $a_{ij} = 0$ for all $i \in S$ and $j \in S'$. It means that the products, which belong to set S are not used for production of any product from the set $S^{'}$.

In other words if the matrix is decomposable then there is a subset of indices that by simultaneous renumeration of rows and columns we can find $S = \{1, ..., k\}$ and $S^{'} = \{k+1, ..., n\}$ such that

$$A = \begin{array}{c} \\ S \\ S' \end{array} \begin{array}{c} \begin{array}{cc} S & S' \end{array} \\ \left[\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array}\right] \end{array} \tag{2.2}$$

where $A_{11} : \mathbb{R}^k \to \mathbb{R}^k,\ A_{22} : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$. The matrix A is indecomposable if representation (2.2) is impossible by any simultaneous rows and columns renumeration.

For indecomposable matrix production of any product $1 \leq j \leq n$ require product $1 \leq i \leq n$ directly ($a_{ij} > 0$) or indirectly, i.e. for any i and j such that $a_{ij} = 0$ there is a set of indices $i = i_1, ..., i_{m-1}, i_m = j$, that $a_{i_s, i_{s+1}} > 0, s = 1, 2, ..., m-1$.

The following theorem (see, for example, [2], [8], [10]) plays an important role on the input-output model.

Theorem 1 (Frobenius-Perron). *Any non-negative indecomposable matrix A has a real positive dominant eigenvalue λ_A, i.e. for any eigenvalue λ of matrix A, the following inequality $|\lambda| \leq \lambda_A$ holds and the correspondent to λ_A eigenvector $x_A \in \mathbb{R}^n_{++}$.*

We are ready to formulate (see, for example, [2], [8], [9], [16], [19]) the necessary and sufficient condition for Leontief's model to be productive .

Theorem 2 (Leontief). *For non-negative indecomposable consumption matrix A the input-output model (2.1) is productive iff $\lambda_A < 1$.*

It follows from $\lambda_A < 1$ and Theorem 1 that for all eigenvalues λ of A we have $|\lambda| < 1$, therefore the matrix $B = \|b_{ij}\|$ has the following representation

$$B = (I - A)^{-1} = I + A + A^2 + A^3 + ...$$

For non-negative indecomposable matrix A we have

$$B - A = I + A^2 + A^3 + ... >> 0,$$

which means that total consumption b_{ij} of product $1 \leq i \leq n$ require to produce one item of product $1 \leq j \leq n$ is always greater than direct consumption a_{ij}.

There is another sufficient condition for Leontief's model to be productive.(see, for example, [9], [16], [17], [19])

Let $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n_{++}$ be the price vector, components of which are prices for one item of correspondent product. If for all $1 \leq i \leq n$ products but one we have balance, i.e. there is $1 \leq j \leq n$ that

$$\lambda_j > (\lambda^T A)_j \text{ and } \lambda_i = (\lambda^T A)_i, i \neq j,$$

then the economy is productive, i.e. for any given $c \in \mathbb{R}^n_+$ the system (1.1) has a positive solution and for any given $q \in \mathbb{R}^n_+$ the system (1.2) has a positive solution as well.

In what follows we will consider a non-negative indecomposable matrices A with $\lambda_A < 1$. Therefore the inverse matrix

$$B = (I - A)^{-1} \tag{2.3}$$

exists and it is positive (see, for example, [16]).

As we mentioned already using the dual LP pair (1.3)-(1.4) to find the optimal production and optimal prices does not produce results different from IO.

In order to find optimal production consistent with the production cost and optimal consumption consistent with the consumption prices we need a new tool, which we consider in the following section.

3. Nonlinear Input-Output Equilibrium

It follows from (1.5) that the production vector $x^* \in \mathbb{R}^n_+$ minimizes the total production cost and at the same time is such that the consumption $c(\lambda^*) \in \mathbb{R}^n_+$ defined by the price vector $\lambda^* \in \mathbb{R}^n_+$ is satisfied.

It follows from (1.6) that the price vector λ^* for goods maximizes the total consumption and at the same time guarantees that the price of each unit of goods does not exceeds the production cost given by vector $p(x^*)$.

We would like to emphasize that due to (1.5) the production cost $p(x^*)$ is in agreement with the optimal production vector x^* and due to (1.6) the consumption $c(\lambda^*)$ is in agreement with the optimal price vector λ^*.

Therefore our main concern is the existence of NIOE $y^* = (x^*, \lambda^*)$.

We assume at this point that both p and c are strongly monotone operators, i.e. there is $\alpha > 0$ and $\beta > 0$ that

$$(p(x_1) - p(x_2), x_1 - x_2) \geq \alpha \|x_1 - x_2\|^2, \ \forall x_1, x_2 \in \mathbb{R}^n_+ \tag{3.1}$$

and

$$(c(\lambda_1) - c(\lambda_2), \lambda_1 - \lambda_2) \leq -\beta \|\lambda_1 - \lambda_2\|^2, \ \forall \lambda_1, \lambda_2 \in \mathbb{R}^n_+ \tag{3.2}$$

Assumption (3.1) implies that the production increase of any goods when the production of the rest are fixed leads to the cost per item increase. Moreover the lower bound of the margin is $\alpha > 0$.

Assumption (3.2) implies that the price increase for an item of any product, when the prices for the rest are fixed, leads to consumption decrease of such product. Moreover, the margine has a negative upper bound $-\beta < 0$.

In order to prove NIOE existence we first show that finding NIOE y^* is equivalent to solving a particular variational inequality (VI) on $\Omega = \mathbb{R}^n_+ \times \mathbb{R}^n_+$.

Theorem 3. *For $y^* = (x^*; \lambda^*) \in \Omega$ to be a solution of (1.5) and (1.6) it is necessary and sufficient for y^* to be a solution of the following VI*

$$(g(y^*), y - y^*) \leq 0, \forall y \in \Omega, \tag{3.3}$$

where the operator $g : \Omega \to \mathbb{R}^{2n}$ is given by the following formula

$$g(y) = ((I - A)^T \lambda - p(x); c(\lambda) - (I - A)x)) \tag{3.4}$$

Proof. If $y^* = (x^*, \lambda^*)$ solves (1.5) and (1.6), then y^* is a saddle point of the Lagrangian

$$L(y^*; X, \Lambda) = (p(x^*), X) - (\Lambda, (I - A)X - c(\lambda^*)), \tag{3.5}$$

i.e.

$$y^* \in \operatorname*{Argmin}_{X \in \mathbb{R}^n_+} \max_{\Lambda \in \mathbb{R}^n_+} L(y^*; X, \Lambda)$$

In other words,

$$x^* \in \text{Argmin}\{L(y^*; X, \lambda^*) | X \in \mathbb{R}^n_+\} =$$

$$= \text{Argmin}\{(p(x^*) - (I - A)^T \lambda^*, X) | X \in \mathbb{R}^n_+\}$$

$$= \text{Argmax}\{(I - A)^T \lambda^* - p(x^*), X) | X \in \mathbb{R}^n_+\}. \tag{3.6}$$

Therefore

$$(I - A)^T \lambda^* - p(x^*) \leq 0. \tag{3.7}$$

On the other hand

$$\lambda^* \in \text{Argmax}\{L(y^*; x^*, \Lambda) | \Lambda \in \mathbb{R}^n_+\}$$

$$= \text{Argmax}\{(c(\lambda^*) - (I - A)x^*, \Lambda) | \Lambda \in \mathbb{R}^n_+\}. \tag{3.8}$$

Therefore

$$c(\lambda^*) - (I - A)x^* \leq 0. \tag{3.9}$$

Keeping in mind the complementarity condition $(g(y^*), y^*) = 0$ for the dual LP pair (1.5)-(1.6) we conclude that (3.3) holds for any $y = (x, \lambda) \in \Omega$, i.e. $y^* = (x^*; \lambda^*)$ solves VI (3.3).

Now let assume that $\bar{y} \in \Omega$ is the solution of VI (3.3), then

$$(g(\bar{y}), y) \leq (g(\bar{y}), \bar{y}), \ \forall y \in \Omega. \tag{3.10}$$

It means that $g(\bar{y}) \leq 0$, otherwise the left hand side can be made as large as one wants by taking the correspondent component of vector y large enough. Therefore we have

$$(I - A)^T \bar{\lambda} - p(\bar{x}) \leq 0, \ \ \bar{x} \geq 0 \tag{3.11}$$

and

$$c(\bar{\lambda}) - (I - A)\bar{x} \leq 0, \ \ \bar{\lambda} \geq 0. \tag{3.12}$$

So $\bar{x}$ is a feasible solution for LP

$$\min\{(p(\bar{x}), x) | (I - A)x \geq c(\bar{x}), x \in \mathbb{R}^n_+\} \tag{3.13}$$

and $\bar{\lambda}$ is a feasible solution for the following dual LP

$$\max\{(c(\bar{x}), \lambda) | (I - A)^T \lambda \leq p(\bar{x}) | \lambda \in \mathbb{R}^n_+\}. \tag{3.14}$$

From (3.10) for $y = 0 \in \mathbb{R}^{2n}$ we have

$$(g(\bar{y}), \bar{y}) \geq 0,$$

which together with (3.11)-(3.12) leads to

$$(g(\bar{y}), \bar{y}) = 0. \tag{3.15}$$

Therefore $(\bar{x}; \bar{\lambda})$ is primal and dual feasible solution for the dual LP pair (3.13) - (3.14) and the complementarity condition (3.15) is satisfied.

Therefore $\bar{x}$ solves (3.13) and $\bar{\lambda}$ solves (3.14), hence $\bar{y} = y^*$ and the proof is completed □

Finding a saddle point of the Lagrangian (3.5), on the other hand, is equivalent to finding an equilibrium of two person concave game with payoff function

$$\varphi_1(y;X,\lambda) = -L(y;X,\lambda) = ((I-A)^T\lambda - p(x), X) + (\lambda, c(\lambda))$$

and strategy $X \in \mathbb{R}^n_+$ for the first player and payoff function

$$\varphi_2(y;x,\Lambda) = L(y;x,\Lambda) = (c(\lambda) - (I-A)x, \Lambda) + (p(x), x)$$

and strategy $\Lambda \in \mathbb{R}^n_+$ for the second player.

Let's consider the normalized payoff function $\Phi : \Omega \times \Omega \to R$, which is given by the following formula

$$\Phi(y;Y) = \varphi_1(y;X,\lambda) + \varphi_2(y;x,\Lambda),$$

then finding $y^* = (x^*;\lambda^*) \in \Omega$ is equivalent to finding a fixed point of the following map

$$y \to \omega(y) = \text{Argmax}\{\Phi(y;Y) | Y \in \Omega\}. \tag{3.16}$$

The normalized payoff function $\Phi(y;Y)$ is linear in $Y \in \Omega$ for any given $y \in \Omega$, but Ω is unbounded, which makes impossible the use of Kakutani's fixed point Theorem to prove the existence of $y^* \in \Omega : y^* \in \omega(y^*)$.

The operator $g : \Omega \to \mathbb{R}^{2n}$ will be called pseudo-gradient because of the following formula

$$g(y) = \nabla_Y \Phi(y;Y) = ((I-A)^T\lambda - p(x); c(\lambda) - (I-A)x) \tag{3.17}$$

We will prove the existence of the NIOE (1.5)-(1.6) later.

First, let us prove the following Lemma.

Lemma 1. *If operators p and c are strongly monotone, i.e. (3.1)-(3.2) are satisfied, then the operator $g : \mathbb{R}^{2n}_+ \to \mathbb{R}^{2n}$, given by (3.17), is strongly monotone as well, i.e. there is $\gamma > 0$ that*

$$(g(y_1) - g(y_2), y_1 - y_2) \le -\gamma \|y_1 - y_2\|^2 \quad \forall y_1, y_2 \in \Omega. \tag{3.18}$$

Proof. Let consider $y_1 = (x_1;\lambda_1)$ and $y_2 = (x_2;\lambda_2) \in \Omega$, then

$$(g(y_1) - g(y_2), y_1 - y_2) = ((I-A)^T\lambda_1 - p(x_1) - (I-A)^T\lambda_2 + p(x_2), x_1 - x_2) +$$

$$+ (c(\lambda_1) - (I-A)x_1 - c(\lambda_2) + (I-A)x_2, \lambda_1 - \lambda_2) =$$

$$((I-A)^T(\lambda_1 - \lambda_2), x_1 - x_2) - (p(x_1) - p(x_2), x_1 - x_2) +$$

$$+ (c(\lambda_1) - c(\lambda_2), \lambda_1 - \lambda_2) - ((I-A)(x_1 - x_2), \lambda_1 - \lambda_2)$$

$$\le -\alpha\|x_1 - x_2\|^2 - \beta\|\lambda_1 - \lambda_2\|^2.$$

Therefore for $\gamma = \min\{\alpha, \beta\}$ we have (3.18).

We are ready to prove the existence of NIOE $y^* = (x^*;\lambda^*)$ in (1.5)-(1.6).

Theorem 4. *If p and c are continuous operators and the strong monotonicity assumption (3.1)-(3.2) hold, then the NIOE $y^* = (x^*, \lambda^*)$ defined by (1.5)-(1.6) exists and it is unique.*

Proof. Let consider $y_0 \in \mathbb{R}^{2n}_{++}$ such that $\|y_0\| \leq 1$ and large enough number $M > 0$. We replace Ω by $\Omega_M = \{y \in \Omega : \|y - y_0\| \leq M\}$. Now the set Ω_M is bounded and convex and so is the set

$$\omega_M(y) = \text{Argmax}\{\Phi(y, Y)|Y \in \Omega_M\}, \tag{3.19}$$

because for any given $y \in \Omega$ the function $\Phi(y, Y)$ is linear in Y and Ω_M is convex and bounded.

Moreover, due to continuity p and c the map $y \to \omega_M(y)$ is upper semi-continuous.

Therefore Kakutani's Theorem can be applied, i.e. there is $y^*_M \in \Omega_M$ such that $y^*_M \in \omega_M(y^*_M)$ (see, for example, [16]). Now we will show that the constraint $\|y - y_0\| \leq M$ in (3.19) is irrelevant. Let us consider (3.18) with $y_1 = y_0$ and $y_2 = y^*_M$. We obtain

$$\gamma\|y_0 - y^*_M\|^2 \leq (g(y^*_M) - g(y_0), y_0 - y^*_M) = \tag{3.20}$$

$$= (g(y^*_M), y_0 - y^*_M) + (g(y_0), y^*_M - y_0)).$$

From $y^*_M \in \omega_M(y^*_M)$ and (3.17) follows that y^*_M is solution of VI

$$(g(y), Y - y) \leq 0, \ \forall Y \in \Omega_M$$

i.e.

$$(g(y^*_M), y_0 - y^*_M) \leq 0.$$

Therefore from (3.20) we have

$$\gamma\|y_0 - y^*_M\|^2 \leq (g(y_0), y^*_M - y_0).$$

Using Couchy-Scwarz inequality we obtain $\|y_0 - y^*_M\| \leq \gamma^{-1}\|g(y_0)\|$, therefore

$$\|y^*_M\| \leq \|y_0\| + \|y_0 - y^*_M\| \leq 1 + \gamma^{-1}\|g(y_0)\|.$$

So, for M large enough the constraint $\|y_0 - y\| \leq M$ can not be active, therefore it is irrelevant in (3.19). Hence, $y^* = y^*_M$ solves the VI (3.3), which is equivalent to finding NIOE from (1.5)-(1.6).

The uniqueness follows from (3.18). In fact, assuming that there is two vectors y^* and $\bar{y}$, which solve VI (3.3) we obtain

$$(g(y^*), \bar{y} - y^*) \leq 0 \text{ and } (g(\bar{y}), y^* - \bar{y}) \leq 0$$

or

$$(g(y^*) - g(\bar{y}), y^* - \bar{y}) \geq 0,$$

which contradicts (3.18) with $y_1 = y^*$ and $y_2 = \bar{y}$, i.e. y^* is unique. □

The strong monotonicity assumption (3.1) and (3.2) are just sufficient for existence of NIOE. In the rest of the paper we assume the existence NIOE.

In what is following we replace the global strong monotonicity assumptions (3.1)-(3.2) for both operators p and c by less restrictive assumptions of local strong monotonicity only at the NIOE $y^* = (x^*; \lambda^*)$.

We assume the existence of $\alpha > 0$ and $\beta > 0$ such that

$$(p(x) - p(x^*), x - x^*) \geq \alpha\|x - x^*\|^2, \ \forall x \in \mathbb{R}^n_+ \tag{3.21}$$

$$(c(\lambda) - c(x^*), \lambda - \lambda^*) \leq -\beta\|\lambda - \lambda^*\|^2, \forall \lambda \in \mathbb{R}^n_+ \tag{3.22}$$

holds.

In the next section we also replace the global Lipschitz continuity of p and c by corresponding assumptions at the NIOE $y^* = (x^*; \lambda^*)$.

$$\|p(x) - p(x^*)\| \leq L_p \|x - x^*\|, \ \ \forall x \in \mathbb{R}^n_+ \tag{3.23}$$

and

$$\|c(\lambda) - c(\lambda^*)\| \leq L_c \|\lambda - \lambda^*\|, \forall \lambda \in \mathbb{R}^n_+. \tag{3.24}$$

We will say that both production p and consumption c operators are well defined if (3.21)-(3.24) hold.

The assumption (3.21) means that the production cost operator p is sensitive to the production change only at the equilibrium. The assumption (3.22) means that the consumption operator c is sensitive to the prices change only at the equilibrium. Lipschitz condition (3.23)-(3.24) means that the production and consumption is under control in the neigbourhood of the NIOE $y^* = (x^*; \lambda^*)$.

4. Pseudo-Gradient Projection Method

The VI (3.3) has a simple feasible set $\Omega = \mathbb{R}^n_+ \times \mathbb{R}^n_+$, projection on which is a very simple operation. Therefore we will concentrate on two methods, for which projection on Ω is the main operation per step.

For a vector $u \in \mathbb{R}^q$ the projection on R^q_+ is given by formula

$$v = P_{R^q_+}(u) = [u]_+ = ([u_1]_+, ..., [u_q]_+)^T,$$

where for $1 \leq i \leq q$ we have

$$[u_i]_+ = \begin{cases} u_i, \, u_i \geq 0 \\ 0, \, u_i < 0 \end{cases}$$

The operator $P_\Omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n_+ \times \mathbb{R}^n_+$ we define by formula

$$P_\Omega(y) = ([x]_+, [\lambda]_+),$$

where $y = (x; \lambda), \ x \in \mathbb{R}^n, \lambda \in \mathbb{R}^n$.

The following two well known properties of the projection operator P_Ω will be used later.

First, the operator P_Ω is not expansive, i.e. for any $u_1 \in \mathbb{R}^{2n}, u_2 \in \mathbb{R}^{2n}$ we have

$$\|P_\Omega(u_1) - P_\Omega(u_2)\| \leq \|u_1 - u_2\| \tag{4.1}$$

Second, the vector u^* solves the VI (3.3) iff for any $t > 0$ the vector u^* is a fixed point of the map $P_Q(I + tg) : \Omega \to \Omega$, i.e.

$$u^* = P_\Omega(u^* + tg(u^*)). \tag{4.2}$$

Let us recall that the VI operator $g : \Omega \to \mathbb{R}^{2n}$ is defined by the following formula

$$g(y) = ((I - A)^T \lambda - p(x); c(\lambda) - (I - A)x).$$

We are ready to describe the Pseudo-Gradient Projection (PGP) method for finding NIOE $y^* \in \Omega$.

Let $y^0 = (x^0; \lambda^0) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$be the starting point and let the pair $y_s = (x_s; \lambda_s)$ has been found already.

The PGP method finds the next approximation by the following formula

$$y_{s+1} = P_\Omega(y_s + tg(y_s)). \tag{4.3}$$

We will specify the step length $t > 0$ later.

In other words, the PGP method simultaneously updates production vector x_s and the price vector λ_s by the following formulas

$$x_{j,s+1} = [x_{j,s} + t((I-A)^T\lambda_s - p(x_s))_j]_+, \quad 1 \le j \le n \tag{4.4}$$

$$\lambda_{i,s+1} = [\lambda_{i,s} + t(c(\lambda_s) - (I-A)x_s)_i]_+, \quad 1 \le i \le n. \tag{4.5}$$

Therefore PGP is primal-dual decomposition method, which allows computing the primal and dual vectors independently and simultaneously.

Formulas (4.4)-(4.5) can be viewed as a pricing mechanism for eatablishing NIOE.

From (4.4) follows: if the price $((I-A)^T\lambda_s)_j$ for one unit of product $1 \le j \le n$ is greater then the production cost $p_j(x_s)$ then the production of $x_{j,s}$ has to be increased.
On the other hand if production cost $p_j(x_s)$ is greater than the price $((I-A)^T\lambda_s)_j$ of one unit of product $1 \le j \le n$ than the production $x_{j,s}$ has to be reduced.

From (4.5) follows: if the consumption $c_i(\lambda_s)$ of product $1 \le i \le n$ is greater then the production $((I-A)x_s)_i$ then the price $\lambda_{i,s}$ has to be increased. If the consumption $c_i(\lambda_s)$ is less than the production $((I-A)x_s)$ then the price $\lambda_{i,s}$ has to be reduced.

The PGP method (4.5)-(4.6) can be viewed as a projected explicit Euler method for solving the following system of ordinary differential equations

$$\frac{dx}{dt} = (I-A)^T\lambda - p(x)$$

$$\frac{d\lambda}{dt} = c(\lambda) - (I-A)x$$

with $x(0) = x_0$ and $\lambda(0) = \lambda_0$.

To prove convergence of the PGP method (4.3) we need the following lemma similar to Lemma 1.

LEMMA 2. *If operators p and c are strongly monotone at x^* and λ^*, i.e. (3.21)-(3.22) holds, then the operator $g : \Omega \to \mathbb{R}^{2n}$ given by (3.4) is strongly monotone at y^* i.e. for $\gamma = \min\{\alpha, \beta\} > 0$ the following bound holds.*

$$(g(y) - g(y^*), y - y^*) \le -\gamma\|y - y^*\|^2, \forall y \in \Omega. \tag{4.6}$$

The proof is similar to the proof of Lemma 1.

LEMMA 3. *If operators p and c satisfy local Lipschitz condition (3.23)-(3.24), then operator $g : \Omega \to \mathbb{R}^{2n}$ given by (3.4) satisfies local Lipschitz condition at y^*, i.e. there is $L > 0$ such that*

$$\|g(y) - g(y^*)\| \le L\|y - y^*\|, \ \forall y \in \Omega. \tag{4.7}$$

For the proof of Lemma 3 and the upper bound for L see Appendix.

REMARK 1. *Let us assume that for a given $x \in \mathbb{R}^n_+$ computing $p(x)$ and for a given $\lambda \in \mathbb{R}^n_+$ computing $c(\lambda)$ does not require more then $O(n^2)$ operations. It is true, for example, if $c(\lambda) = \nabla(\frac{1}{2}\lambda^T C\lambda + d^T\lambda)$ and $p(x) = \nabla(\frac{1}{2}x^T Px + q^T x)$, where $C : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric negative semidefinite matrix and $P : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric positive semidefinite matrix. Then each PGP step does not require more* *then $O(n^2)$ operations, because the rest is matrix $(I - A)$ by vector multiplication twice.*

Let $\varkappa = \gamma L^{-1}$ be the condition number of the VI operator $g : \Omega \to \mathbb{R}^{2n}$. The following theorem establishes global Q-linear convergence rate and complexity of the PGP method (4.3). The proof is similar to the proof of Theorem 3 in [21]. We will sketch the proof for completeness.

THEOREM 5. *If operators p and c are well defined (i.e. (3.21)-(3.24) holds) then:*

(1) *for any $0 < t < 2\gamma L^{-2}$ the PGP method (4.3) globally converges to NIOE $y^* = (x^*; \lambda^*)$ with Q-linear rate and the ratio $0 < q(t) = (1 - 2t\gamma + t^2L^2)^{\frac{1}{2}} < 1$, i.e.*

$$\|y_{s+1} - y^*\| \le q(t)\|y_s - y^*\| \tag{4.8}$$

(2) *for $t = \gamma L^{-2} = \min\{q(t)|t > 0\}$ the following bound holds*

$$\|y_{s+1} - y^*\| \le (1 - \varkappa^2)^{\frac{1}{2}}\|y_s - y^*\| \tag{4.9}$$

(3) *the PGP complexity is given by the following bound*

$$comp(PGP) = O(n^2\varkappa^{-2}\ln\varepsilon^{-1}), \tag{4.10}$$

where $\varepsilon > 0$ is the given accuracy.

PROOF. From (4.1) - (4.3) follows

$$\|y_{s+1} - y^*\|^2 = \|P_\Omega(y_s + tg(y_s)) - P_\Omega(y^* + tg(y^*))\|^2 \le$$

$$\le \|y_s + tg(y_s) - y^* - tg(y^*)\|^2 = \tag{4.11}$$

$$= \|y_s - y^*\|^2 + 2t(y_s - y^*, g(y_s) - g(y^*)) + t^2\|g(y_s) - g(y^*)\|^2$$

From Lemma 2, Lemma 3 and (4.11) follows

$$\|y_{s+1} - y^*\| \le \|y_s - y^*\|(1 - 2t\gamma + t^2L^2)^{\frac{1}{2}},$$

therefore for any $0 < t < 2\gamma L^{-2}$ we have $0 < q(t) = (1-2t\gamma+t^2L^2)^{\frac{1}{2}} < 1$.

For $t = \gamma L^{-2} = \text{Argmin}\{q(t)|t > 0\}$ we have

$$q = q(\gamma L^{-2}) = (1 - (\gamma L^{-1})^2)^{\frac{1}{2}} = (1 - \varkappa^2)^{\frac{1}{2}}.$$

Let $0 < \varepsilon << 1$ be the required accuracy . From (4.8) follows that it takes $O((\ln q)^{-1}\ln\varepsilon)$ steps to find an ε- approximation for NIOE $y^* = (x^*, \lambda^*)$. Each step requires $O(n^2)$ operations, therefore the complexity bound for the PGP method (4.3) is

$$\mathcal{N} = O\left(n^2\frac{\ln\varepsilon^{-1}}{\ln q^{-1}}\right).$$

From $(\ln q^{-1})^{-1} = (-\frac{1}{2}\ln(1 - \varkappa^2))^{-1}$ and $\ln(1 + x) \le x, \;\; \forall x > -1$ follows $(\ln q^{-1})^{-1} \le 2\varkappa^{-2}$, which leads to the bound (4.10). □

In case $\gamma = \min\{\alpha, \beta\} = 0$. Theorem 5 can't guarantee even convergence of the PGP method. We need another tool for solving VI (3.3).

In the following section we apply Extra Pseudo-Gradient (EPG) method for solving VI (3.3). The EPG method was first introduced by G. Korepelevich [15] in the 70s for finding saddle points. Since then the EPG became a popular tool for solving VI (see [5]-[7],[12]-[14], [21]).

We first assume that the operators p and c are monotone and satisfy Lipschitz condition, so is the operator g, i.e. for $\forall y_1, y_2 \in \Omega$ we have

$$(g(y_1) - g(y_2), y_1 - y_2) \leq 0 \tag{4.12}$$

and

$$\|g(y_1) - g(y_2)\| \leq L\|y_1 - y_2\|. \tag{4.13}$$

The monotonicity property (4.12) follows from monotonicity properties of operators p and c (see Lemma 1). The Lipschitz condition (4.13) follows from correspondent condition of c and p.

5. Extra Pseudo-Gradient method for finding NIOE

Application of G.Korpelevich extra-gradient method for solving VI (3.3) leads to the following method for finding NIOE.

Each step of the EPG method consists of two phases: in the prediction phase we predict the production vector $\hat{x} \in \mathbb{R}^n_+$ and the price vector $\hat{\lambda} \in \mathbb{R}^n_+$, in the corrector phase we correct the production and the price vectors by using the predicted cost per unit vector $p(\hat{x})$ and the predicted consumption vectors $c(\hat{\lambda})$.

Let $y_0 = (x_0, \lambda_0) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$ be the starting point. We assume that the approximation $y_s = (x_s, \lambda_s)$ has been found already.

The predictor phase finds

$$\hat{y_s} = P_\Omega(y_s + tg(y_s)) = [y_s + tg(y_s)]_+. \tag{5.1}$$

The corrector phase finds the new approximation

$$y_{s+1} = P_\Omega(y_s + tg(\hat{y}_s)) = [y_s + tg(\hat{y}_s)]_+. \tag{5.2}$$

The step length $t > 0$ will be specified later.

In other words, the EPG method first predicts the production vector

$$\hat{x}_s = [x_s + t((I - A)^T\lambda_s - p(x_s))]_+ \tag{5.3}$$

and the price vector

$$\hat{\lambda}_s = [\lambda_s + t(c(\lambda_s) - (I - A)x_s)]_+. \tag{5.4}$$

Then EPG finds the new production vector

$$x_{s+1} = [x_s + t((I - A)^T\hat{\lambda}_s - p(\hat{x}_s))]_+ \tag{5.5}$$

and the new price vector

$$\lambda_{s+1} = [\lambda_s + t(c(\hat{\lambda}_s) - (I - A)\hat{x}_s)]_+. \tag{5.6}$$

The meaning of formulas (5.3)-(5.6) is similar to the meaning of formulas (4.4)-(4.5).

The EPG method (5.1)-(5.2), in fact, is a pricing mechanism for establishing NIOE. The following Theorem establishes the convergence of EPG.

Theorem 6. *If p and c are monotone operators and Lipschitz condition (4.13) is satisfied, then for any $t \in (0, (\sqrt{2}L)^{-1})$ the EPG method (5.1)-(5.2) generates a converging sequence $\{y_s\}_{s=0}^\infty$ and $\lim_{s\to\infty} y_s = y^*$.*

(1) Proof. It follows from (5.1)-(5.2), non-expansiveness of P_Q and Lipschitz condition (4.13) that

$$\|y_{s+1}-\hat{y}_s\| = \|P_\Omega(y_s+tg(\hat{y}_s)) - P_\Omega(y_s+tg(y_s))\|$$

$$\le t\|g(\hat{y}_s)-g(y_s)\| \le tL\|\hat{y}_s-y_s\|. \tag{5.7}$$

From (5.1) follows

$$(tg(y_s)+(y_s-\hat{y}_s), y-\hat{y}_s) \le 0, \ \ \forall y \in \Omega,$$

therefore by taking $y = y_{s+1}$, we obtain

$$(tg(y_s)+(y_s-\hat{y}_s, y_{s+1}-\hat{y}_s) \le 0$$

or

$$(y_s-\hat{y}_s, y_{s+1}-\hat{y}_s) + t(g(\hat{y}_s), y_{s+1}-\hat{y}_s) -$$

$$-t(g(\hat{y}_s)-g(y_s), y_{s+1}-\hat{y}_s) \le 0. \tag{5.8}$$

It follows from (5.7) and Lipschitz condition (4.13)

$$(g(\hat{y}_s)-g(y_s), y_{s+1}-\hat{y}_s) \le \|g(\hat{y}_s)-g(y_s)|\|y_{s+1}-\hat{y}_s\| \le tL^2\|\hat{y}_s-y_s\|^2. \tag{5.9}$$

From (5.8) and (5.9) we obtain

$$(y_s-\hat{y}_s, y_{s+1}-\hat{y}_s) + t(g(\hat{y}_s), y_{s+1}-\hat{y}_s) - (tL)^2\|\hat{y}_s-y_s\|^2 \le 0. \tag{5.10}$$

From (5.2) follows

$$(tg(\hat{y}_s)+y_s-y_{s+1}, y-y_{s+1}) \le 0, \ \ \forall y \in \Omega. \tag{5.11}$$

Therefore for $y = y^*$ we have

$$(y_s-y_{s+1}+tg(\hat{y}_s), y^*-y_{s+1}) \le 0. \tag{5.12}$$

Also from $(g(y^*), y-y^*) \le 0, \ \forall y \in \Omega$ we obtain $(g(y^*), \hat{y}_s-y^*) \le 0$, so for $\forall t > 0$ we have

$$t(-g(y^*), y^*-\hat{y}_s) \le 0. \tag{5.13}$$

By adding (5.10),(5.12) and (5.13) and using the monotonicity of g, i.e.

$$(g(\hat{y}_s)-g(y^*), y^*-\hat{y}_s) \ge 0$$

we obtain

$$2(y_s-y_{s+1}, y^*-y_{s+1}) + 2(y_s-\hat{y}_s, y_{s+1}-\hat{y}_s) - 2(tL)^2\|\hat{y}_s-y_s\|^2 \le 0. \tag{5.14}$$

Using the following itentity

$$2(u-v, w-v) = \|u-v\|^2 + \|v-w\|^2 - \|u-w\|^2 \tag{5.15}$$

twice first with $u = y_s, v = y_{s+1}$ and $w = y^*$ and second with $u = y_s, v = \hat{y}_s$ and $w = y_{s+1}$ we obtain

$$2(y_s-y_{s+1}, y^*-y_{s+1}) = \|y_s-y_{s+1}\|^2 + \|y_{s+1}-y^*\|^2 - \|y_s-y^*\|^2 \tag{5.16}$$

and

$$2(y_s-\hat{y}_s, y_{s+1}-\hat{y}_s) = \|y_s-\hat{y}_s\|^2 + \|\hat{y}_s-y_{s+1}\|^2 - \|y_s-y_{s+1}\|^2. \tag{5.17}$$

From (5.14), (5.16) and (5.17) follows

$$\|y_{s+1}-y^*\|^2 + (1-2(tL)^2)\|y_s-\hat{y}_s\|^2 + \|\hat{y}_s-y_{s+1}\|^2 \le \|y_s-y^*\|^2. \tag{5.18}$$

Summing up (5.18) from $s = 0$ to $s = N$ we obtain

$$\|y_{N+1} - y^*\|^2 + (1 - 2(tL)^2) \sum_{s=0}^{N} \|y_s - \hat{y}_s\|^2 + + \sum_{s=0}^{N} \|\hat{y}_s - y_{s+1}\|^2 \le \|y_0 - y^*\|^2. \tag{5.19}$$

It follows from (5.19) that for $0 < t \le \frac{1}{\sqrt{2}L}$ we have $\sum_{s=0}^{\infty} \|y_s - \hat{y}_s\|^2 < \infty$ and $\sum_{s=0}^{\infty} \|\hat{y}_s - y_{s+1}\|^2 < \infty$, which means that

$$(a) \ \lim_{s\to\infty} \|y_s - \hat{y}_s\| \to 0 \quad \text{and } (b) \ \lim_{s\to\infty} \|\hat{y}_s - y_{s+1}\| \to 0. \tag{5.20}$$

Also from (5.18) follows that

$$\|y_{s+1} - y^*\| \le \|y_s - y^*\| \ \ \forall s \ge 0. \tag{5.21}$$

Thus $\{y_s\}_{s=0}^{\infty}$ is bounded sequence, therefore there is a converging sub-sequence $\{y_{s_k}\}_{s_k>1}^{\infty} : \lim_{k\to\infty} y_{s_k} = \bar{y}$. It follows from (5.20a) that $\lim_{k\to\infty} \hat{y}_{s_k} = \bar{y}$ and from (5.20b) follows $\lim_{k\to\infty} y_{s_k+1} = \bar{y}$. □

From the continuity of the operator g we have

$$\bar{y} = \lim_{k\to\infty} y_{s_k+1} = \lim_{k\to\infty} [y_{s_k} + tg(\hat{y}_{s_k})]_+ = [\bar{y} + tg(\bar{y})]_+.$$

From (4.2) we obtain that $\bar{y} = y^*$, which together (5.21) leads to $\lim_{s\to\infty} y_s = y^*$.

6. Convergence rate and complexity of the EPG method.

In this section we establish the convergence rate and complexity of the EPG method under Lipschitz continuity and local strong monotonicity of both operators $p : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and $c : \mathbb{R}^n_+ \to \mathbb{R}^n_+$. The proof has some similarities with correspondent proof in [21], but the complexity bound is better due to the structure of matrix D. We have (7.1) without rescaling matrix D.

The EPG require two projection on Ω while PGP requires only one such projection. Keeping in mind that the most costly operation per projection is matrix by vector multiplication the EPG does not increase the complexity bound $O(n^2)$ per step.

On the other hand, EPG converges in case of just monotonicity of g, i.e. when $\gamma = 0$. Moreover EPG has much better then PGP complexity bound in case of local strong monotonicity, in particular, when the condition number $0 < \varkappa < 1$ is small.

To establish the convergence rate and complexity of EPG we will need two inequalities, which are following from the local strong monotonicity of g.

First, by adding (5.10) and (5.12) we obtain

$$(y_s - y_{s+1}, y^* - y_{s+1}) + t(g(\hat{y}_s), y^* - \hat{y}_s) + (y_s - \hat{y}_s, y_{s+1} - \hat{y}_s) - -(tL)^2 \|y_s - \hat{y}_s\|^2 \le 0. \tag{6.1}$$

Second, it follows from (4.6) that

$$(g(y), y - y^*) - (g(y^*), y - y^*) \le -\gamma \|y - y^*\|^2. \tag{6.2}$$

Keeping in mind $(g(y^*), y - y^*) \le 0, \ \ \forall y \in \Omega$ from (6.2) we obtain

$$(g(y), y^* - y) \geq \gamma \|y - y^*\|^2, \ \forall y \in \Omega. \tag{6.3}$$

THEOREM 7. *If for both p and c operators the local strong monotonicity (3.21)-(3.22) holds and Lipschitz condition (4.13) satisfies, then*

(1) *there exists* $0 < q(t) < 1, \ \forall t \in (0, (\sqrt{2}L)^{-1})$ *that*

$$\|y_{s+1} - y^*\| \leq q(t)\|y_s - y^*\| \tag{6.4}$$

(2)

$$\|y_{s+1} - y^*\| \leq \sqrt{1 - 0.5\varkappa}\|y_s - y^*\|, \ \forall \varkappa \in [0, 0.5] \tag{6.5}$$

(3)

$$Comp(EPG) \leq O(n^2 \varkappa^{-1} \ln \varepsilon^{-1}) \tag{6.6}$$

where $\varepsilon > 0$ *is the required accuracy.*

PROOF. (1) From (6.3) with $y = \hat{y}_s$ we have

$$(g(\hat{y}_s), y^* - \hat{y}_s) \geq \gamma \|\hat{y}_s - y^*\|^2.$$

Therefore (6.1) we can rewrite as follows.

$$2(y_s - y_{s+1}, y^* - y_{s+1}) + 2(y_s - \hat{y}_s, y_{s+1} - \hat{y}_s) \tag{6.7}$$
$$+2\gamma t\|\hat{y}_s - y^*\|^2 - 2(tL)^2\|\hat{y}_s - y_s\|^2 \leq 0.$$

Using identity (5.15), first, with $u = y_s, v = y_{s+1}, w = y^*$ and second with $u = y_s, v = y_{s+1}, w = \hat{y_s}$ from (6.7) we obtain

$$\|y_{s+1} - y^*\|^2 + \|\hat{y}_s - y_{s+1}\|^2 + (1 - 2(tL)^2)\|y_s - \hat{y}_s\|^2 \tag{6.8}$$
$$+2\gamma t\|\hat{y}_s - y^*\|^2 \leq \|y_s - y^*\|^2.$$

Using

$$\|\hat{y}_s - y^*\|^2 = \|\hat{y}_s - y_s\|^2 + 2(\hat{y} - y_s, y_s - y^*) + \|y_s - y^*\|^2$$

from (6.8) we obtain

$$\|y_{s+1} - y^*\|^2 + \|\hat{y}_s - y_{s+1}\|^2 + (1 - 2(tL)^2)\|\hat{y}_s - y_s\|^2 +$$
$$+2\gamma t\|\hat{y}_s - y_s\|^2 + 4\gamma t(\hat{y}_s - y_s, y_s - y^*) + 2\gamma t\|y_s - y^*\|^2 \leq \|y_s - y^*\|^2$$

or

$$\|y_{s+1} - y^*\|^2 + \|\hat{y}_s - y_{s+1}\|^2 + (1 + 2\gamma t - 2(tL)^2)\|\hat{y}_s - y_s\|^2 +$$
$$+4\gamma t(\hat{y}_s - y_s, y_s - y^*) \leq (1 - 2\gamma t)\|y_s - y^*\|^2. \tag{6.9}$$

Let $\mu(t) = 1 + 2\gamma t - 2(tL)^2$, then the third and the fourth term of the left hand side can be rewritten as follows

$$\|\sqrt{\mu(t)}(\hat{y} - y_s) + 2(y_s - y^*)\frac{\gamma t}{\sqrt{\mu(t)}}\|^2 - \frac{4(\gamma t)^2}{\mu(t)}\|y_s - y^*\|^2.$$

Therefore from (6.9) we obtain bound (6.4)

$$\|y_{s+1} - y^*\| \leq q(t)\|y_s - y^*\|$$

with $q(t) = (1 - 2\gamma t + 4(\gamma t)^2(\mu(t))^{-1})^{\frac{1}{2}}$. It is easy to see that for $t \in (0, (\sqrt{2}L)^{-1})$ we have $0 < q(t) < 1$.

(2) For $\varkappa = \gamma L^{-1}$ and $t = (2L)^{-1}$ we obtain

$$q((2L)^{-1}) = [(1+\varkappa)(1+2\varkappa)^{-1}]^{\frac{1}{2}}.$$

For $0 \le \varkappa \le 0.5$ we have

$$q = q((2L)^{-1}) \le \sqrt{1-0.5\varkappa},$$

therefore the bound (6.5) holds

(3) It follows from (6.5) that for a given accuracy $0 < \varepsilon << 1$ the EPG method requires

$$s = O\left(\frac{\ln \varepsilon^{-1}}{\ln q^{-1}}\right)$$

steps to get $y_s : \|y_s - y^*\| \le \varepsilon$. Then we have

$$(\ln q^{-1})^{-1} = \left(-\frac{1}{2}\ln(1-0.5\varkappa))^{-1}\right).$$

Using $\ln(1+x) \le x,\ \forall x > -1$ we obtain $\ln(1-0.5\varkappa) \le -0.5\varkappa$ or

$$-0.5\ln(1-0.5\varkappa) \ge 0.25\varkappa.$$

Therefore $(\ln q^{-1})^{-1} \le 4\varkappa^{-1}$. Keeping in mind that each EPG step requires $O(n^2)$ operation we obtain bound (6.6) □

REMARK 2. *For small $0 < \varkappa < 1$ the EPG complesity bound (6.6)is much better than PGP bound (4.10), therefore the necessity to project on Ω twice per step is easily compensated by faster convergence of the EPG.*

In case of $1 > \varkappa > 0.5$ and large n, however, the PGP could be more efficient.

7. Appendix

Let us estimate the Lipschitz constant $L > 0$ in (4.13), which plays an important role in both PGP and EPG methods.

First of all due to the productivity of the matrix A we have $\max_{1\le j\le n}\sum_{i=1}^n a_{ij} \le 1$ and $\max_{1\le i\le n}\sum_{i=1}^n a_{ij} \le 1$.

Therefore, for matrix

$$D = I - A = \begin{pmatrix} 1-a_{11} & \cdots & -a_{1n} & \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & 1-a_{nn} & \end{pmatrix} = \begin{pmatrix} d_{11} & \cdots & d_{1n} & \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nn} & \end{pmatrix}$$

we obtain

$$\|D\|_I = \max_{1\le j\le n}\sum_{i=1}^n |a_{ij}| \le 2,\ \|D\|_{II} = \max_{1\le i\le n}\sum_{j=1}^n |d_{ij}| \le 2. \tag{7.1}$$

Let consider operators $p : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and $c : \mathbb{R}^n_+ \to \mathbb{R}^n_+$. We assume that for both operators Lipschitz condition is satisfied for each component of correspodent vector-function, i.e. for $x_1, x_2 \in \mathbb{R}^n_+$ we have

$$|p_j(x_1) - p_j(x_2)| \le L_{p,j}\|x_1 - x_2\|, 1 \le j \le n \tag{7.2}$$

and for $\lambda_1, \lambda_2 \in \mathbb{R}^n_+$ we have

$$|c_i(\lambda_1) - c_i(\lambda_2)| \le L_{c,i}\|\lambda_1 - \lambda_2\|,\ \ 1 \le i \le n. \tag{7.3}$$

It follows from (7.2) that for any $x_1, x_2 \in \mathbb{R}^n_+$

$$\|p(x_1) - p(x_2)\| = \sqrt{\sum_{j=1}^{n} (p_j(x_1) - p_j(x_2))^2} \le \sqrt{\sum_{i=1}^{n} L^2_{p,j} \|x_1 - x_2\|^2} \le$$

$$L_p \sqrt{n\|x_1 - x_2\|^2} = L_p \sqrt{n} \|x_1 - x_2\|, \tag{7.4}$$

where $L_p = \max_{1 \le j \le n} L_{p,j}$.

Similarly, we obtain

$$\|c(\lambda_1) - c(\lambda_2)\| = \sqrt{\sum (c_i(\lambda_1) - c_i(\lambda_2)^2} \le$$

$$\sqrt{\sum_{i=1}^{n} L^2_{c,i} \|\lambda_1 - \lambda_2\|^2} \le L_c \sqrt{n} \|\lambda_1 - \lambda_2\| \tag{7.5}$$

where $L_c = \max_{1 \le i \le n} L_{c,i}$.

We are ready to find the upper bound for L in (4.13). Using (3.4) for any pair $y_1, y_2 \in \Omega = \mathbb{R}^n_+ \times \mathbb{R}^n_+$ we have

$$\|g(y_1) - g(y_2)\| \le \|(I - A)^T \lambda_1 - p(x_1) - (I - A)^T \lambda_2 + p(x_2)\|$$

$$+ \|c(\lambda_1) - (I - A)x_1 - c(\lambda_2) + (I - A)x_2\|$$

$$\le \|p(x_1) - p(x_2)\| + \|(I - A)^T\| \|\lambda_1 - \lambda_2\| +$$

$$+ \|c(\lambda_1) - c(\lambda_2)\| + \|(I - A)^T\| \|x_1 - x_2\| \le$$

$$(L_p \sqrt{n} + \|I - A\|) \|x_1 - x_2\| + (L_c \sqrt{n} + \|(I - A)^T\|) \|\lambda_1 - \lambda_2\|. \tag{7.6}$$

For $D = I - A$ and $D^T = (I - A)^T$ we have

$$\|D\| = \sqrt{\lambda_{max}(D^T D)} \ \text{ and } \ \|D^T\| = \sqrt{\lambda_{max}(D D^T)}$$

Keeping in mind (7.1) we obtain (see [10])

$$\|D\| \le \sqrt{n} \|D\|_I = 2\sqrt{n}, \quad \|D^T\| \le \sqrt{n} \|D^T\|_{II} = 2\sqrt{n}. \tag{7.7}$$

Using (7.4),(7.5) and (7.8) from (7.6) we obtain

$$\|g(y_1) - g(y_2)\| \le \sqrt{n}(L_p + 2) \|x_1 - x_2\| +$$

$$+ \sqrt{n}(L_c + 2) \|\lambda_1 - \lambda_2\| \le \sqrt{2} \hat{L} (\sqrt{n} + 2) \|y_1 - y_2\|$$

where $\hat{L} = \max\{L_p, L_c\}$. Therefore for the Lipschitz constant in (4.13) we have the following bound

$$L \le \sqrt{2} \hat{L} (\sqrt{n} + 2) = O(\sqrt{n}). \tag{7.8}$$

8. Concluding remarks

The NIOE adds some new features to the classical IO model. In particular, it finds the output consistent with production cost and consumption consistent with prices for goods.

At the NIOE the output minimizes the total production cost while the price vector maximizes the total consumption.

It looks like solving problems (1.5)-(1.6) is much more difficult then solving IO systems (1.1)-(1.2). In reality both the PGP and EPG are very simple methods, which require neither solving subproblems nor systems of equations. At most each step requires few matrix by vector multiplications.

Therefore in same instances finding NIOE can be even less expensive than solving large systems of linear equations.

In particular, it follows from the bound (6.6) and (7.8) that EPG complexity is $O(n^{2.5} \ln \varepsilon^{-1})$ for any given fixed $\gamma > 0$.

Therefore for very large n the EPG method can be used when solving linear systems (2.1)-(2.2) is very difficult or even impossible.

Full primal- dual decomposition for both methods allows computing the primal and dual vectors in parallel.

Along with parallel technique for matrix by vector multiplication (see, for example, [22] and references therein) both PGP and EPG can be used for developing numerically efficient methods for large scale NIOE.

Three main issues are left for further research.

First, numerical realization and correspondent software based on both PGP and EPG methods need to be developed.

Second, numerical experiments with real data have to be performed.

Third, economic analyses of the numerical results and its comparison with results obtained by IO has to be conducted.

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