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A Modified Barrier-Augmented Lagrangian Method for Constrained Minimization

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Abstract. We present and analyze an interior-exterior augmented Lagrangian method for solving constrained optimization problems with both inequality and equality constraints. This method, the modified barrier—augmented Lagrangian (MBAL) method, is a combination of the modified barrier and the augmented Lagrangian methods. It is based on the MBAL function, which treats inequality constraints with a modified barrier term and equalities with an augmented Lagrangian term. The MBAL method alternatively minimizes the MBAL function in the primal space and updates the Lagrange multipliers. For a large enough fixed barrier-penalty parameter the MBAL method is shown to converge Q-linearly under the standard second-order optimality conditions. Q-superlinear convergence can be achieved by increasing the barrier-penalty parameter after each Lagrange multiplier update. We consider a dual problem that is based on the MBAL function. We prove a basic duality theorem for it and show that it has several important properties that fail to hold for the dual based on the classical Lagrangian.

1. Introduction

In this paper we develop a new method for solving constrained nonlinear optimization problems involving both inequality and equality constraints. Our method, the modified barrier—augmented Lagrangian (MBAL) method, is a combination of the augmented Lagrangian method for equality constraints of Hestenes [10] and Powell [14] and the modified barrier function (MBF) method of Polyak [12]. Variants of the latter method have been considered by Breitfeld and Shanno [4] and Conn et al. [5]. Since the modified barrier function can be viewed as an interior augmented Lagrangian, the MBAL method can be viewed as an interior-exterior augmented Lagrangian method.

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The idea of combining a barrier function and a penalty function approach to solve constrained optimization problems with both inequality and equality constraints was suggested nearly thirty years ago by Fiacco and McCormick ([18]). Their interior-exterior method uses a classical barrier function to treat the inequality constraints and a penalty function to treat the equalities. In their approach the barrier-penalty parameter is the only means by which the computational process can be controlled. Therefore to guarantee convergence this parameter has to be increased to infinity, which leads to numerical problems due to the ill-conditioning of the Hessian of the barrier-penalty function.

The MBAL function used in the MBAL method eliminates major drawbacks of the barrier-penalty function while retaining its best features. In contrast with the barrier-penalty function, the MBAL function exists at the solution and inherits the smoothness of the objective and constraint functions in a neighborhood of that solution. Under standard second-order optimality conditions the MBAL function has a unique global minimizer for any vector of positive Lagrange multipliers for the inequality constraints and any vector of Lagrange multipliers for the equality constraints when the barrier-penalty parameter is large enough. The dual function based on the MBAL function is smooth and the dual problem based on it has several important properties.

Finally, the MBAL method converges Q-linearly for a fixed barrier-penalty parameter at a rate that can be made as fast as desired by using a large enough value for the fixed parameter. Since this parameter does not have to be increased to infinity to ensure convergence as in the classical barrier-penalty approach the condition number of the Hessian of the MBAL function remains bounded allowing Newton's method to be applicable for minimizing the MBAL function in a larger region. Q-superlinear convergence can be achieved by choosing a sequence of penalty-barrier parameters tending to infinity.

Although the augmented Lagrangian method was originally designed for problems with equality constraints, it was extended to handle inequality constraints by Rockafellar [15]. This extention has been well studied and shares many of the positive qualities of the MBAL method (e.g., see Chapter 5 in [3]); however the augmented Lagrangian for inequality constraints is differentiable only once even if the objective and constraint functions possess higher differentiability. It is also possible to extend the modified barrier function method to handle equality constraints by replacing each equality by two inequalities. This approach not only increases the number of constraints but also, more importantly, introduces an ill-conditioned barrier for the equalities; hence it is not recommended.

The main contribution of this paper is the demonstration that the MBAL method has a rate of convergence that is up to Q-superlinear like the augmented Lagrangian and modified barrier function methods, using proof techniques similar to those used in [3] and [12]. The MBAL method converges globaly in the dual space. It converges globaly in the primal space if global unconstrained optimization is performed on every iteration. Another contribution is the development of some duality results based on the MBAL function.

Our paper is organized as follows. In the next section the general nonlinear programming problem and the basic assumptions under which our convergence results hold are stated. The MBAL function, on which our method is based, is introduced in Section 3. We describe the MBAL method in Section 4 and present convergence and rate of convergence results for it and discuss some aspects concerning the practical implementation of the method in

Section 5. Proofs of the main results presented in Section 5 are given in Section 6. Duality results based on the MBAL function are discussed in Section 7. An appendix containing two technical lemmas concludes the paper.

2. Problem statement and basic assumptions

In this paper we consider the following general nonlinear programming problem:

$$\begin{array}{rll} \text{minimize} & f_0(x) & \\ \text{s.t.} & f_i(x) \geq 0 & i = 1, \ldots, p, \\ & g_j(x) = 0 & j = 1, \ldots, q, \end{array} \tag{1}$$

where $x \in \mathbf{R}^n$, and we assume that $f_0, f_1, \ldots, f_p$ and $g_1, \ldots, g_q$ are C^2 functions from $\mathbf{R}^n \to \mathbf{R}$.

The classical Lagrangian for this problem is:

$$L(x, u, v) = f_0(x) - u^T f(x) - v^T g(x),$$

where $u \in \mathbf{R}^p_+$, $v \in \mathbf{R}^q$ and $f(x) = (f_1(x), \ldots, f_p(x))$ and $g(x) = (g_1(x), \ldots, g_q(x))$ are column vectors. $\mathbf{R}^p_+$ denotes the nonnegative orthant of $\mathbf{R}^p$, and $\mathbf{R}^p_{++}$ its interior.

Let x^* be a strict local minimum of problem (1) and $I^* = \{i : f_i(x^*) = 0\} = \{1, \ldots, r\}$ be the set of indices of the inequality constraints that are active at that point. Throughout this paper we assume that the standard second-order sufficient conditions for an isolated local minimum hold at x^*, namely:

- C1. The gradients $\nabla f_i(x^*)$, $i = 1, \ldots, r$ and $\nabla g_j(x^*)$, $j = 1, \ldots, q$ are **linearly independent**; hence, there exists a unique Lagrange multiplier vector $w^* = (u^*, v^*) \in \mathbf{R}^p_+ \times \mathbf{R}^q$ such that:

$$\nabla_x L(x^*, u^*, v^*) = \nabla f_0(x^*) - \sum_{i=1}^{p} u_i^* \nabla f_i(x^*) - \sum_{j=1}^{q} v_j^* \nabla g_j(x^*) = 0. \tag{2}$$

- C2. The Hessian of the Lagrangian $L(x, u, v)$ with respect to x at (x^*, u^*, v^*),

$$\nabla^2_{xx} L(x^*, u^*, v^*) = \nabla^2 f_0(x^*) - \sum_{i=1}^{p} u_i^* \nabla^2 f_i(x^*) - \sum_{j=1}^{q} v_j^* \nabla^2 g_j(x^*),$$

is **positive definite** on the affine subspace tangent to the feasible set at x^*; i.e.,

$$y^T \nabla^2_{xx} L(x^*, u^*, v^*) y > 0, \tag{3}$$

for all $y \in Y \subset \mathbf{R}^n$, where

$$Y = \{y : y^T \nabla f_i(x^*) = 0, \ i = 1, \ldots, r, \ y^T \nabla g_j(x^*) = 0, \ j = 1, \ldots, q\}.$$

- C3. **Strong complementary slackness** holds for the inequality constraints; i.e.

$$u_i^* f_i(x^*) = 0,\ i = 1, \dots, p \tag{4}$$

$$u_i^* > 0,\ i = 1, \dots, r;\ \ f_i(x^*) > 0,\ i = r+1, \dots, p. \tag{5}$$

3. The modified barrier-augmented Lagrangian function

We define the modified barrier—augmented Lagrangian function $F(x, u, v, k)\colon \mathbf{R}^n \times \mathbf{R}_+^p \times \mathbf{R}^q \times \mathbf{R}_{++} \to \mathbf{R}$ by the formula:

$$F(x,u,v,k) = \begin{cases} f_0(x) \ - k^{-1}\sum_{i=1}^{p} u_i \ln(kf_i(x)+1) & \\ \qquad - \sum_{j=1}^{q} v_j g_j(x) + k/2 \sum_{j=1}^{q} g_j^2(x), & \text{if } x \in \operatorname{int}\Omega_k \\ \infty, & \text{if } x \notin \operatorname{int}\Omega_k, \end{cases}$$

where $\Omega_k = \{x : f_i(x) \geq -k^{-1}, i = 1, \dots, p\}$. $F(x, u, v, k)$ contains modified barrier terms for the inequality constraints in (1) and augmented Lagrangian terms for the equality constraints in (1). If the complementary slackness condition (4) holds at the point (x^*, u^*, v^*), then for any $k > 0$:

(P1) $\quad F(x^*, u^*, v^*, k) = f_0(x^*)$,

(P2) $\quad \nabla_x F(x^*, u^*, v^*, k)$

$$= \nabla f_0(x^*) - \sum_{i=1}^{p} \frac{u_i^*}{kf_i(x^*)+1} \nabla f_i(x^*) - \sum_{j=1}^{q} (v_j^* - kg_j(x^*)) \nabla g(x^*)$$

$$= \nabla f_0(x^*) - \sum_{i=1}^{p} u_i^* \nabla f_i(x^*) - \sum_{j=1}^{q} v_j^* \nabla g_j(x^*) = 0,$$

(P3) $\quad \nabla_{xx}^2 F(x^*, u^*, v^*, k)$

$$= \nabla_{xx}^2 L(x^*, u^*, v^*) + k\nabla f^T(x^*) U^* \nabla f(x^*) + k\nabla g^T(x^*) \nabla g(x^*).$$

Here $\nabla f(x)$ and $\nabla g(x)$ are the Jacobian matrices of the vector functions f and g respectively, and U^* is a diagonal matrix with diagonal entries $u_i^*,\ i = 1, \dots, p$.

In contrast with the classical barrier-penalty function, the MBAL function is defined at x^* and has the same order of smoothness as the functions $f_i(x),\ i = 1, \dots, p$, and $g_j(x),\ j = 1, \dots, q$ in a neighborhood of (x^*, u^*, v^*). Moreover, from property P3, and Lemma A.1 in the Appendix, we have

Theorem 3.1. *Suppose that the second-order optimality conditions* C1–C3 *hold at* x^*. *Then there exists a* $k_0 > 0$ *such that, for every* $k \geq k_0$, *the matrix* $\nabla_{xx}^2 F(x^*, u^*, v^*, k)$ *is positive definite; i.e.,* $F(x, u^*, v^*, k)$ *is strongly convex at* x^*.

4. The modified barrier-augmented Lagrangian method

If the second-order sufficient conditions C1–C3 hold at a solution x^* to problem (1), it follows from properties P1–P3 and Theorem 3.1 that to solve problem (1)—i.e., to find a global minimizer x^*—one need only find the unconstrained minimum of the smooth strongly convex function $F(x, u^*, v^*, k)$ in a neighborhood of x^*, for $k > 0$, fixed but large enough.

We shall show that, if the vector of Lagrange multipliers $w = (u, v) \in \mathbf{R}^p_{++} \times \mathbf{R}^q$ is close enough to $w^* = (u^*, v^*)$,

$$\hat{x} = \hat{x}(u, v, k) = \operatorname{argmin}\{F(x, u, v, k) \mid x \in \mathbf{R}^n\}$$

is a good approximation to x^*. It turns out that the minimizer $\hat{x}$ can be used to improve the approximation $w = (u, v)$ to the optimal Lagrange multipliers $w^* = (u^*, v^*)$ provided that the fixed penalty-barrier parameter $k > 0$ is sufficiently large. Consequently, by alternatively minimizing $F(x, u, v, k)$ and updating the Lagrange multipliers (u, v), we are able to solve problem (1) starting from *any* initial point (x, w), $x \in \operatorname{int} \Omega_k$ and $w \equiv (u, v) \in \mathbf{R}^p_{++} \times \mathbf{R}^q$, for a choice of the penalty-barrier parameter k that is large enough; i.e., we have a globally convergent method.

Let us consider this in more detail. Assuming that the unconstrained minimizer $\hat{x} = \hat{x}(u, v, k)$ exists, we have

$$\begin{aligned} &\nabla_x F(\hat{x}, u, v, k) \\ &\quad = \nabla f_0(\hat{x}) - \sum_{i=1}^{p} \frac{u_i}{(k f_i(\hat{x}) + 1)} \nabla f_i(\hat{x}) - \sum_{j=1}^{q} (v_j - k g_j(\hat{x})) \nabla g_j(\hat{x}) = 0. \end{aligned} \tag{6}$$

After defining new Lagrange multipliers by the formulas

$$\hat{u}_i = \frac{u_i}{k f_i(\hat{x}) + 1}, \quad i = 1, \ldots, p, \tag{7}$$

$$\hat{v}_j = v_j - k g_j(\hat{x}), \quad l = 1, \ldots, q, \tag{8}$$

we can rewrite (6) as

$$\begin{aligned} \nabla_x F(\hat{x}, \hat{u}, \hat{v}, k) &= \nabla f_0(\hat{x}) - \sum_{i=1}^{p} \hat{u}_i \nabla f_i(\hat{x}) - \sum_{j=1}^{q} \hat{v}_j \nabla g_j(\hat{x}) \\ &= \nabla_x L(\hat{x}, \hat{u}, \hat{v}) = 0. \end{aligned} \tag{9}$$

Therefore, the unconstrained minimizer $\hat{x}$ of $F(x, u, v, k)$ is a stationary point of the classical Lagrangian $L(x, \hat{u}, \hat{v})$. In the convex case $\hat{x}$ is the minimizer of this Lagrangian.

Let $\hat{w}(u, v, k) = (\hat{u}, \hat{v}) = \hat{w}$. First, $\hat{w}(w^*, k) = w^*$ for any fixed $k > 0$; i.e., w^* is a fixed point of the mapping $w \to \hat{w}(w, k)$. Second, it will be shown later that

$$\|\hat{w} - w^*\| \le C k^{-1} \|w - w^*\| \tag{10}$$

and

$$\|\hat{x} - x^*\| \le C k^{-1} \|w - w^*\|, \tag{11}$$

where $C > 0$ is independent of $k > 0$. Here and throughout the paper the norm $\|\cdot\|$ denotes the l_∞-norm $\|\cdot\|_\infty$. In other words, by finding an unconstrained minimizer for the MBAL function $F(x, w, k)$ in x and updating the Lagrange multipliers we can shrink the distance between (x, w) and (x^*, w^*) by a factor that can be made arbitrarily small by increasing the barrier-penalty parameter $k > 0$.

The above reasoning gives rise to what we call the MBAL method, whose s-th iteration is:

Given a penalty parameter $k^s > 0$ *and estimates* (x^s, u^s, v^s) *of* (x^*, u^*, v^*), *where* $u^s \in \mathbf{R}^p_{++}$, $v^s \in \mathbf{R}^q$ *and* $x^s \in \operatorname{int}\Omega_k$, *compute*

$$x^{s+1} = \operatorname{argmin}\{F(x, u^s, v^s, k^s) : x \in \mathbf{R}^n\}, \tag{12}$$

$$u_i^{s+1} = \frac{u_i^s}{k^s f_i(x^{s+1}) + 1}, \quad i = 1, \ldots, p, \tag{13}$$

$$v_j^{s+1} = v_j^s - k^s g_j(x^{s+1}), \quad j = 1, \ldots, q. \tag{14}$$

In the next section we prove that the MBAL method with a fixed barrier-penalty parameter k^s converges linearly whenever the second-order optimality conditions are fulfilled and $k^s > 0$ is large enough. Although this result holds for fixed k^s, and ill-conditioning of the Hessian $\nabla^2_{xx}F(x, w, k^s)$ is less of a problem as $x^s \to x^*$ in this case, it is useful in practice to increase the k^s from step to step to obtain superlinear convergence. However, one has to be careful to make sure that the current primal minimizer x^{s+1} lies in $\Omega_{k^{s+1}}$. One way to get around this difficulty is to replace the logarithmic barrier term by a quadratic barrier when the argument of the barrier term is smaller than some given value (e.g., see [2]). A truncated modified barrier method incorporating this approach has been used successfully to solve large-scale truss-topology design problems (see [1] and [2]) and encouraging numerical results have been reported recently by Breitfeld and Shanno [4] and by Nash et al. [11] for different versions of this method. Alternative ways of controlling the increase in k^s and obtaining a good rate of convergence for shifted barriers are described in [5].

We now introduce some notation, that is used in the rest of the paper. Let a and b be vectors in $\mathbf{R}^p$. Then a^Tb denotes the usual scalar product. ab denotes the vector with components a_ib_i, a/b or ab^{-1} denotes the vector with components a_i/b_i and $\ln(a)$ denotes the vector with components $\ln(a_i)$. That is, we denote the componentwise operations on vectors as if the vectors are scalars. We also use $a_{(r)}$ and $a_{(p-r)}$ to denote the vectors that consist of the first r and the last $p-r$ components, respectively, of the vector a, and e to denote a vector of ones of appropriate dimension.

5. Convergence results

In the proof of our main theorems we need $\frac{\|w-w^*\|}{k}$ to be small, and hence we require the neighborhood of $w^* \equiv (u^*, v^*)$ to depend on k. Therefore, for a given $\delta > 0$ and $k_0 > 0$, we define the following set

$$D(w^*, \delta, k_0) = \{(w, k) = (u, v, k) : \|w - w^*\| \leq \delta k,\ u_{(r)} > 0, u_{(p-r)} \geq 0,\ k \geq k_0\},$$

which is a truncated cone. We can now state our main results.

Theorem 5.1. *Suppose that conditions* C1–C3 *hold at a strict local minimum* x^* *of problem* (1). *Then there exist* $k_0 > 0$ *and* $\delta > 0$ *such that for any triple* $(u, v, k) = (w, k) \in D(w^*, \delta, k_0,)$, *with* $\|w\|$ *bounded, the following statements are true*:

- $F(x, u, v, k)$ *has a unique minimizer* $\hat{x} \equiv \hat{x}(u, v, k)$ *with respect to* x *within some open ball centered at* x^*; *i.e.,*

$$\nabla_x F(\hat{x}, u, v, k) = 0$$

  and $F(x, u, v, k)$ *is strongly convex in a neighborhood of* $\hat{x}$.
- *For the triple* $(\hat{x}, \hat{u}, \hat{v}) : \hat{u} = u(kf(\hat{x}) + 1)^{-1},\ \hat{v} = v - kg(\hat{x})$ *the following estimates hold*

$$\begin{aligned} \|\hat{x} - x^*\| &\le \frac{C}{k}\|w - w^*\|, \\ \|\hat{w} - w^*\| &\le \frac{C}{k}\|w - w^*\|, \end{aligned} \tag{15}$$

  where $\hat{w} = (\hat{u}, \hat{v})$ *and* $C > 0$ *is independent of* k.

To prove the global convergence of the MBAL method we require the following additional assumption:

A1. There exists a $k_0 > 0$ such that, for all fixed $u \ge 0$ and v and all finite α, the level sets

$$L_\alpha(u, v, k_0) = \{x \in \mathbf{R}^n \mid F(x, u, v, k_0) \le \alpha\}$$

are bounded.

Theorem 5.2. *Suppose that conditions* C1–C3 *hold at a* (*global*) *solution* x^* *to problem* (1) *and assumption* A1 *holds. Then there is a* $k_0 > 0$ *large enough so that for any triple* $(u, v, k) \in D(w^*, \sigma, k_0)$ *the vector* $\hat{x}$ *in Theorem* 5.1 *is the global minimizer of* $F(x, u, v, k)$.

Proofs of the above results are given in the next section. First, however, we discuss Assumption A1 and show that it is not unduly restrictive and consider some of the important consequences and extensions of these results.

It is easy to verify that Assumption A1 is satisfied in the following cases given that $f_0, f_1, \dots, f_p$ and $g_1, \dots, g_q$ are C^2 functions from $\mathbf{R}^n$ to $\mathbf{R}$.

(i) Problem (1) is a convex programming problem (i.e., $f_0(x)$ and $-f_i(x)$, $i = 1, \dots, p$, are convex functions and g_j, $j = 1, \dots, q$, are linear functions) whose set of optimal solutions is bounded.
(ii) There exists a k_0 such that the set Ω_{k_0} is compact.

(iii) There exists a $k_0 > 0$ and $\tau > 0$ such that

$$\max\left\{ \max_{1\le i\le p} \left\{ f_i(x) \mid x \in \Omega_{k_0} \right\} \right\} \le \tau$$

and $f_0(x)$ is bounded below on Ω_{k_0}.

The above cases demonstrate that Assumption A1 is likely to be satisfied by most nonlinear programming problems that arise in practice. It also follows from this assumption that, for all $k \ge k_0$ and any fixed $u \in \mathbf{R}^p_+$ and $v \in \mathbf{R}^q$ and all finite α, the level sets $L_\alpha(u, v, k)$ are bounded. To prove this we note that it follows from the mean value formula $\ln(1+t) = t/(1+\tau)$, where τ is some scalar between 0 and t, that $\ln(1+t) \ge t/(1+t)$ for all $t > -1$. Hence

$$\begin{aligned}\frac{d}{dk}F(x,u,v,k) = \frac{1}{k^2}\sum_{i=1}^{p} u_i[\ln(kf_i(x)+1) - kf_i(x)/(kf_i(x)+1)]\\ + \frac{1}{2}\sum_{j=1}^{q} g_j^2(x) \ge 0,\end{aligned}$$

which implies that for all $k \ge k_0$, $F(x,u,v,k) \ge F(x,u,v,k_0)$ and $L_\alpha(u,v,k) \subseteq L_\alpha(u,v,k_0)$.

Theorems 5.1 and 5.2 have several important consequences which we now discuss.

The MBAL method reduces the solution of problem (1) to a sequence of unconstrained minimization problems. Moreover, even though problem (1) is nonconvex, Theorem 5.1 shows that these unconstrained minimization problems are smooth and the MBAL function is strongly convex in a neighborhood of each unconstrained minimizer as long as the standard second-order optimality conditions are satisfied and the finite barrier-penalty parameter k is chosen large enough.

A strong point of the MBAL method as stated by (12)–(14) is that it determines a global minimizer x^* of problem (1). However, this requires finding the unconstrained global minimizer of the MBAL function on each iteration, which is a weakness because this function is not, in general, convex. After the first iteration, this global minimization is not that difficult if a large enough barrier-penalty parameter is chosen, since by Theorem 5.1, x^s, the global minimizer found on the last iteration, is then close to x^* and hence, to x^{s+1}, and the MBAL function is strongly convex in a neighborhood of x^{s+1}.

It is easy to see from Theorems 5.1 and 5.2 that if the MBAL method starts at $(u^0, v^0, k^0) \in D(w^*, \delta, k_0)$, then all iterates $(u^s, v^s, k^s) \in D(w^*, \delta, k_0)$ and w^s remains bounded as long as k^s is chosen so that $C/k^s \le 1$. Thus, one has complete freedom in choosing the initial estimates $u^0 \in \mathbf{R}^p_{++}$ and $v^0 \in \mathbf{R}^q$ of the optimal Lagrange multipliers u^* and v^*; i.e., the MBAL method is globally convergent, assuming that the solution to problem (1) is unique. However, if the initial estimates u_0 and v_0 are far from u^* and v^*, one must pay the price of using suitably large barrier-penalty parameters k^s.

If the parameters k^s remain finite, it follows from Theorems 5.1 and 5.2 that the MBAL method converges at least Q-linearly. A superlinear rate of convergence for the MBAL method is achieved if the sequence of barrier-penalty parameters $\{k^s\}$ is chosen so that

$\lim_{s\to\infty} k^s = \infty$. The price paid for this, however, is that the subproblems that need to be solved become increasingly ill-conditioned as in the classical penalty-barrier method.

Instead of a single barrier-penalty parameter k, one can choose a barrier parameter for each inequality constraint and a penalty parameter for each equality constraint, i.e., instead of the MBAL function, one can consider the function

$$F(x, u, v, K) = f_0(x) - \sum_{i=1}^{p} k_i^{-1} u_i \ln(k_i f_i(x) + 1) - \sum_{j=1}^{q} v_j g_j(x) + \sum_{j=1}^{q} \frac{\kappa_j}{2} g_j^2(x),$$

where $K = (k_1, \ldots, k_p, \kappa_1, \ldots, \kappa_q)$. All basic results of Theorems 3.1, 5.1 and 5.2 remain valid as long as $\min\{k_i, \kappa_j\} = k \geq k_0$ and k_0 is large enough.

We note that simple bound constraints on the variables, $a_i \leq x_i \leq b_i$, $i = 1, \ldots, n$ are handled by the MBAL method just like any other inequality constraints, in contrast with approaches like the one proposed in [5] that treat them in a special manner. Interestingly, such constraints have the effect of adding positive diagonal terms

$$d_j = k\Big[u_j^{(1)}(k(b_j - x_j) + 1)^{-2} + u_j^{(2)}(k(x_j - a_j) + 1)^{-2}\Big]$$

to the Hessian of the MBAL function increasing its positive definiteness, where $u_j^{(1)}$ and $u_j^{(2)}$ are the positive Lagrange multipliers corresponding to the jth upper and lower bounds, respectively.

Suppose that in each iteration of the MBAL method (12)–(14) the global minimization of the MBAL function is replaced by a local minimization. It then follows from Theorem 5.1 that such a local version of the MBAL method will be convergent to a strict local minimum x^* of problem (1), provided that the local minimizers x^{s+1} of $F(x, u^s, v^s, k^s)$ computed by the method are those local minimizers of $F(x, u^s, v^s, k^s)$ that are closest to x^*. Fortunately, this will usually happen if the unconstrained minimization routine that is used to compute x^{s+1} is a descent method that is started from the previous local minimizer x^s. If on the other hand, the local minimizers x^{s+1} are not in the neighborhood of the same local minimizer x^* after some iteration, then our convergence analysis does not apply. Similar remarks apply to "global" MBAL method if problem (1) has multiple solutions.

The MBAL method (12)–(14) involves finding the global unconstrained minimizer $\hat{x}$ of the MBAL function on each iteration. As this requires an infinite number of operations in general, the "pure" MBAL method is not practical. Rather, only an approximation $\bar{x}$ to the exact minimizer $\hat{x}$ can be computed in practice. Analogous remarks apply to the "local" version of the MBAL method where $\hat{x}$ is a local unconstrained minimizer of the MBAL function and $\bar{x}$ is an approximation to $\hat{x}$. If $\bar{x}$ is required to satisfy, for some $\gamma > 0$, the inequality

$$\|\nabla_x F(\bar{x}, u, v, k)\| \leq \gamma k^{-1} \|(kf(\bar{x}) + 1)^{-1} u - u\| + \gamma \|g(\bar{x})\|, \tag{16}$$

then an analog of Theorem 5.1 with $\hat{x}$ replaced by $\bar{x}$, $\hat{w}$ by $\bar{w}$ (computed using $\bar{x}$) and C by $C(1 + \gamma)$ can be proved. In particular, using arguments similar to those used to prove Theorem 5 in [12] and Lemma 2 in [13], the following assertion can be proved.

Assertion 5.3. *If the assumptions of Theorem* 5.1 *are satisfied then there exists an* $\bar{x}$ *within some open ball computed at* x^* *that satisfies* (16) *and*

$$\|\bar{x} - x^*\| \leq \frac{C(1+\gamma)}{k}\|w - w^*\|,$$
$$\|\bar{w} - w^*\| \leq \frac{C(1+\gamma)}{k}\|w - w^*\|.$$

6. Proof of Theorems 5.1 and 5.2

Before we prove the Theorem 5.1 let us consider the main ideas behind our proof. One can rewrite (6)–(8) as the following system of equations for $\hat{x}$, $\hat{u}_{(r)}$ and $\hat{v}$:

$$\nabla f_0(\hat{x}) - \sum_{i=1}^{r} \hat{u}_i \nabla f_i(\hat{x}) - \sum_{j=1}^{q} \hat{v}_j \nabla g_j(\hat{x}) - h\big(\hat{x}, u_{(p-r)}, k\big) = 0, \tag{17}$$

where

$$\hat{u}_{(r)} = u\big(kf_{(r)}(\hat{x}) + e_r\big)^{-1}, \tag{18}$$
$$\hat{v} = v - kg(\hat{x}), \tag{19}$$

and

$$h\big(\hat{x}, u_{(p-r)}, k\big) = \sum_{i=r+1}^{p} u_i(kf_i(\hat{x}) + 1)^{-1}\nabla f_i(\hat{x}). \tag{20}$$

It is easy to see that $\hat{x} = x^*$, $\hat{u}_{(r)} = u^*_{(r)}$ and $\hat{v} = v^*$ satisfy this system for any $k > 0$ and $\hat{u}_{(p-r)} = u^*_{(p-r)}$. Moreover, for k_0 large enough we shall show that for any triple $(u, v, k) = (w, k) \in D(w^*, \delta, k_0)$ one can obtain the solution $\hat{x} = \hat{x}(u, v, k)$, $\hat{u}_{(r)} = \hat{u}_{(r)}(u, v, k)$ and $\hat{v} = \hat{v}(u, v, k)$ to the system (17)–(19). Taking into account the smoothness of this solution as a function of $(w, k) = (u, v, k)$, we can compute its Jacobian and prove under the second-order optimality conditions C1–C3 that, for any fixed $k \geq k_0$, there exists $C > 0$, independent of k, such that

$$\max\big\{\|\nabla_w \hat{x}(w, k)\|, \big\|\nabla_w \hat{u}_{(r)}(w, k)\big\|, \|\nabla_w \hat{v}(w, k)\|\big\} \leq C, \tag{21}$$

for any $(w, k) \in D(w^*, \delta, k_0)$. Since $x^* = \hat{x}(w^*, k)$, $u^*_{(r)} = \hat{u}_{(r)}(w^*, k)$ and $v^* = \hat{v}(w^*, k)$, we can then bound $\|\hat{x} - x^*\|$, $\|\hat{u}_{(r)} - u^*_{(r)}\|$ and $\|\hat{v} - v^*\|$ in terms of $\|w - w^*\|$. We then show that $F(x, u, v, k)$ is strongly convex at $\hat{x}$.

Proof of Theorem 5.1: First we have to prove that $F(x, u, v, k)$ has a local minimizer $\hat{x}$ whenever $(w, k) \in D(w^*, \delta, k_0)$. For convenience let us shift the neighborhood of w^* to an appropriate neighborhood of the origin in the dual space by introducing the vector $t = (t^u, t^v) = (w - w^*)k^{-1}$, where

$$t^u = (u - u^*)k^{-1} \quad \text{and} \quad t^v = (v - v^*)k^{-1}. \tag{22}$$

Then in terms of t, the vector of updated Lagrange multipliers corresponding to the active constraints $\hat{u}_{(r)} = (kt^u_{(r)} + u^*_{(r)})(kf_{(r)}(\hat{x}) + e_r)^{-1}$. Therefore, $k^{-1}(kt^u_{(r)} + u^*_{(r)})(kf_{(r)}(x) + e_r)^{-1} - k^{-1}\hat{u}_{(r)} = 0$. Keeping in mind that $u^*_{(p-r)} = 0$, the updated Lagrange multipliers corresponding to the passive constraints

$$\hat{u}_{(p-r)}\big(x, t^u_{(p-r)}, k\big) = kt^u_{(p-r)}\big(kf_{(p-r)}(x) + e_{(p-r)}\big)^{-1}. \tag{23}$$

Let us also define

$$h\big(x, t^u_{(p-r)}, k\big) = \sum_{i=r+1}^{p} \hat{u}_i\big(x, t^u_i, k\big)\nabla f_i(x) = k\sum_{i=r+1}^{p} t^u_i (kf_i(x) + 1)^{-1}\nabla f_i(x).$$

It is clear that the vector-function $h(x, t^u_{(p-r)}, k)$ is smooth up to second-order and

$$\begin{aligned} h(x^*, 0, k) &= 0, & \nabla_x h(x^*, 0, k) &= 0^{n\times n}, \\ \nabla_{\hat{u}_{(r)}} h(x^*, 0, k) &= 0^{n\times r}, & \nabla_{\hat{v}} h(x^*, 0, k) &= 0^{n\times q}. \end{aligned}$$

For $k > 0$, consider the mapping $\Phi_k(x, \hat{u}_{(r)}, \hat{v}, t) : \mathbf{R}^{n+r+p+2q} \to \mathbf{R}^{n+r+q}$:

$$\Phi_k\big(x, \hat{u}_{(r)}, \hat{v}, t\big) = \begin{bmatrix} \nabla f_0(x) - \nabla f_{(r)}(x)^T\hat{u}_{(r)} - \nabla g(x)^T\hat{v} - h\big(x, t^u_{(p-r)}, k\big) \\ k^{-1}\big(kt^u_{(r)} + u^*_{(r)}\big)\big(kf_{(r)}(x) + e_r\big)^{-1} - k^{-1}\hat{u}_{(r)} \\ k^{-1}(kt^v + v^*) - g(x) - k^{-1}\hat{v} \end{bmatrix}$$

corresponding to the system of Eqs. (17)–(19). We now show that $\Phi_k(\cdot)$ satisfies the conditions of the implicit function theorem and apply that theorem to obtain estimates of the proximity of $(\hat{x},\ \hat{u},\ \hat{v})$ to $(x^*,\ u^*,\ v^*)$.

From condition C1 and the formulas for the Lagrange multiplier updates,

$$\Phi_k\big(x^*, u^*_{(r)}, v^*, 0\big) = 0.$$

The Jacobian of $\Phi_k(x, \hat{u}_{(r)}, \hat{v}, t)$ with respect to $(x, \hat{u}_{(r)}, \hat{v})$, at $(x, \hat{u}_{(r)}, \hat{v}, t) = (x^*, u^*_{(r)}, v^*, 0)$,

$$\begin{aligned} \nabla\Phi_k &\equiv \nabla_{(x,\hat{u}_{(r)},\hat{v})}\big(\Phi_k\big(x^*, u^*_{(r)}, v^*, 0\big)\big) \\ &= \begin{bmatrix} \nabla^2_{xx}L(x^*, u^*, v^*) & -\nabla_x f^T_{(r)}(x^*) & -\nabla_x g^T(x^*) \\ -U^*_{(r)}\nabla_x f_{(r)}(x^*) & -k^{-1}I^r & 0^{r\times q} \\ -\nabla_x g(x^*) & 0^{q\times r} & -k^{-1}I^q \end{bmatrix}, \end{aligned} \tag{24}$$

where $U^*_{(r)} = \operatorname{diag}(u^*_i)^r_{i=1}$.

For $k = \infty$ this matrix is nonsingular. This can be proved by trivially extending Lemma 1.27 in [3]. Thus, for k large enough $\nabla\Phi_k$ is nonsingular and there exists a scalar $\rho > 0$

independent of $k \geq k_0$ such that $\|\nabla\Phi_k^{-1}\| \leq \rho$. Consequently, it follows from the second implicit function theorem ([3], p. 12) applied to the system $\Phi_k(x, \hat{u}_{(r)}, \hat{v}, t) = 0$ that there exist smooth vector-functions $x(t)$, $\hat{u}_{(r)}(t)$ and $\hat{v}(t)$ such that $x(0) = x^*$, $\hat{u}_{(r)}(0) = u^*_{(r)}$ and $\hat{v}(0) = v^*$, and in the neighborhood $S(\delta) = \{(t) : \|t\| \leq \delta\}$ and $k \geq k_0$),

$$\Phi_k\big(x(t), \hat{u}_{(r)}(t), \hat{v}(t), t\big) \equiv 0. \tag{25}$$

Therefore, for $x(\cdot) \equiv x(t)$ we obtain from the first n equations of (25)

$$\nabla_x F(x(\cdot)), \hat{u}, \hat{v}, k) = \nabla f_0(x(\cdot)) - \sum_{i=1}^{p} \hat{u}_i(\cdot)\nabla f_i(x(\cdot)) - \sum_{j=1}^{q} \hat{v}_j(\cdot)\nabla g_j(x(\cdot)) = 0; \tag{26}$$

i.e., $x(\cdot)$ is a stationary point of the MBAL function. Before we show that $x(\cdot)$ is a strict local minimum of this function, we first prove the validity the estimates (15).

Let $\hat{u}_{(p-r)}(t, k) = \hat{u}_{p-r}(x(t, k), t^u_{(p-r)}, k)$ and $\alpha = \min\{f_i(x^*) : r + 1 \leq i \leq p\}$. Since as $t \to 0$, $x(t) \to x^*$, $f_i(x(t)) \to f_i(x^*) \geq \alpha > 0$, $i = r + 1, \ldots, p$, it follows that $f_i(x(t)) \geq \alpha/2$ for $i = r + 1, \ldots, p$ and any $t \in S(\delta)$ for $\delta > 0$ small enough. Hence, from (23) and the definition of $t_{(p-r)}$, we have

$$\hat{u}_{(p-r)}(\cdot) \leq \frac{u_{(p-r)} - u^*_{(p-r)}}{k\alpha/2}$$

or, since $u^*_{(p-r)} = 0$,

$$\big\|\hat{u}_{(p-r)}(\cdot) - u^*_{(p-r)}\big\| = \big\|\hat{u}_{(p-r)}(\cdot)\big\| \leq 2\alpha^{-1}k^{-1}\big\|u_{(p-r)} - u^*_{(p-r)}\big\|. \tag{27}$$

From the implicit function theorem we have at $t = 0$

$$\nabla_t\big(x(t), \hat{u}_{(r)}(t), \hat{v}(t)\big)\big|_{t=0} = -\nabla\Phi_k^{-1}\nabla_t\Phi_k\big(x^*, u^*_{(r)}, v^*, t\big)\big|_{t=0}.$$

Hence, for t close enough to 0 it follows from the bound on $\nabla\Phi_k$ (see (24) and the discussion below (24)), that

$$\big\|\big(x(t), \hat{u}_{(r)}(t), \hat{v}(t)\big) - (x^*, u^*, v^*)\big\| \leq 1/\rho\,\big\|\Phi_k\big(x^*, u^*_{(r)}, v^*, t\big) - \Phi_k\big(x^*, u^*_{(r)}, v^*, 0\big)\big\| \tag{28}$$

This estimate can also be derived from an extension of the implicit function theorem given in [7].

From the definition of Φ_k

$$\|\Phi_k(x^*, u^*, v^*, t) - \Phi_k(x^*, u^*, v^*, 0)\| \leq \big\|h(x^*, t_{(p-r)}, k)\big\| + \|t^u\| + \|t^v\|, \tag{29}$$

where

$$\begin{aligned}\left\|h\left(x^*, t_{(p-r)}, k\right)\right\| &= k\left\|\sum_{i=r+1}^{p} t_i^u (kf_i(x)+1)^{-1}\nabla f_i(x^*)\right\| \\ &\leq \left\|\sum_{i=r+1}^{p}\frac{(u_i-u_i^*)}{\alpha k}\nabla f_i(x^*)\right\| \leq \frac{C_1}{k}\left\|u_{(p-r)}-u_{(p-r)}^*\right\| \qquad (30)\end{aligned}$$

for some constant C_1.

Recalling the definition (22) of t^u and t^v and combining (27)–(31) we conclude that, for k large enough, there exists a constant C, independent of k such that

$$\begin{aligned}\|\hat{x}-x^*\| &= \|\hat{x}(w,k)-x^*\| \leq \frac{C}{k}\|w-w^*\|,\\ \|\hat{w}-w^*\| &= \|\hat{w}(w,k)-w^*\| \leq \frac{C}{k}\|w-w^*\|,\end{aligned} \qquad (31)$$

From (26) it follows that $\hat{x}$ is a stationary point of $F(x, u, v, k)$. To show that it is a local minimum we now prove that $F(x, u, v, k)$ is strongly convex in a neighborhood of $\hat{x}$.

$$\begin{aligned}\nabla_{xx}^2 F(\hat{x}, u, v, k) &= \nabla^2 f_0(\hat{x}) - \sum_{i=1}^{p}\hat{u}_i\nabla^2 f_i(\hat{x}) + k\nabla f(\hat{x})^T\hat{U}\hat{D}^{-1}\nabla f(\hat{x})\\ &\quad - \sum_{j=1}^{q}\hat{v}_j\nabla^2 g_j(\hat{x}) + k\nabla g_j(\hat{x})^T\nabla g_j(\hat{x})\\ &= \nabla_{xx}^2 L(\hat{x},\hat{u},\hat{v}) + k\nabla f(\hat{x})^T\hat{U}\hat{D}^{-1}\nabla f(\hat{x}) + k\nabla g(\hat{x})^T\nabla g(\hat{x}),\end{aligned}$$

where $\hat{U} = \operatorname{diag}[u_i(kf_i(\hat{x})+1)^{-1}]_1^p$, $\hat{D} = \operatorname{diag}[(kf_i(\hat{x})+1)]_1^p$. Consider the ith diagonal element of $\hat{U}\hat{D}^{-1}$, where $1 \leq i \leq r$. For all $(w, k) \in D(w^*, \delta, k_0)$, $k_0 > 0$ large enough, and $\delta > 0$ small enough, it follows from the Lipschitz property of the C^2 function $f_i(x)$, in any bounded neighborhood of x^*, from the fact that $f_i(x^*) = 0$ and from (31) that

$$kf_i(\hat{x}) \leq k(f_i(\hat{x}) - f_i(x^*)) \leq kM_i\|\hat{x}-x^*\| \leq M_iC\|w-w^*\|,$$

where M_i is a Lipschitz constant for $f_i(x)$. Since w is in a compact set, $kf_i(\hat{x})+1$ is bounded independent of k; hence, for sufficiently large $k_0 > 0$ and sufficiently small $\delta > 0$, it follows from (31) and the fact that $u_i^* > 0$ that $\hat{u}_i(kf_i(\hat{x})+1)^{-1} \geq \mu_i > 0$, where μ_i is independent of k. Also from (31) we have

$$\nabla_{xx}^2 L(\hat{x},\hat{u},\hat{v}) \cong \nabla_{xx}^2 L(x^*,u^*,v^*), \quad \nabla f(\hat{x}) \cong \nabla f(x^*), \quad \text{and} \quad \nabla g(\hat{x}) \cong \nabla g(x^*).$$

Therefore,

$$\begin{aligned}&\nabla_{xx}^2 F(\hat{x}, u, v, k)\\ &\quad\cong \nabla_{xx}^2 L(x^*,u^*,v^*) + k\nabla f(x^*)^T\hat{U}\hat{D}^{-1}\nabla f(x^*) + k\nabla g(x^*)\nabla g(x^*)^T,\end{aligned}$$

and it then follows from optimality condition C2 and a lemma of Debreu [6; p. 296] (see also Lemma 3 in [9]), that there exists $k_0 > 0$ large enough, such that for all $k \geq k_0$, $\nabla^2_{xx} F(\hat{x}, u, v, k)$ is positive definite; i.e., $F(x, u, v, k)$ is strongly convex at $\hat{x}$, for $\hat{x}$ in a neighborhood of x^*. Hence $\hat{x}$ is a local minimum of $F(x, u, v, k)$. □

Proof of Theorem 5.2: Since from Theorem 5.1, $\hat{x}$ is the minimizer of $F(x, u, v, k)$ in a neighborhood of x^* and $g(x^*) = 0$, $f(x^*) \geq 0$ and $u \geq 0$,

$$\begin{aligned} F(\hat{x}, u, v, k) &\leq F(x^*, u, v, k) \\ &= f_0(x^*) - k^{-1} u^T \ln(k f(x^*) + e) - v^T g(x^*) + \frac{k}{2} \|g(x^*)\|_2^2 \leq f_0(x^*). \end{aligned} \tag{32}$$

Now suppose that there exists a vector $\tilde{x} \in \mathbf{R}^n$ and a number $\tilde{\lambda} > 0$ such that

$$F(\tilde{x}, u, v, k) \leq F(\hat{x}, u, v, k) - \tilde{\lambda}.$$

Then from (32) and the definition of $F(x, u, v, k)$, $\tilde{x} \in \Omega_k$, and we obtain

$$\begin{aligned} F(\tilde{x}, u, v, k) = &\ f_0(\tilde{x}) - k^{-1} \sum_{i=1}^{p} u_i \ln(k f_i(\tilde{x}) + 1) - v^T g(\tilde{x}) + \frac{k}{2} \|g(\tilde{x})\|_2^2 \\ \leq &\ f_0(x^*) - \tilde{\lambda}. \end{aligned}$$

Hence

$$\begin{aligned} f_0(\tilde{x}) &\leq f_0(x^*) + k^{-1} \sum_{i:\, f_i(\tilde{x})>0} u_i \ln(k f_i(\tilde{x}) + 1) + v^T g(\tilde{x}) - \frac{k}{2} \|g(\tilde{x})\|_2^2 - \tilde{\lambda} \\ &= f_0(x^*) + k^{-1} \sum_{i:\, f_i(\tilde{x})>0} u_i \ln(k f_i(\tilde{x}) + 1) - \tilde{\lambda} - \frac{1}{2} \left\| \sqrt{k} g(\tilde{x}) - \frac{v}{\sqrt{k}} \right\|_2^2 \\ &\quad + \frac{1}{2k} \|v\|_2^2. \end{aligned} \tag{33}$$

From the boundedness of u and v (recall that w is bounded), Assumption A1 (recall that the level set $L_{f_0(x^*)}(u, v, k) \subseteq L_{f_0(x^*)}(u, v, k_0)$) and the continuity of f_i, $i = 1, \ldots, p$, we have that

$$k^{-1} \sum_{i:\, f_i(\tilde{x})>0} u_i \ln(k f_i(\tilde{x}) + 1) = O(k^{-1} \ln k)$$

and

$$\frac{1}{2k} \|v\|_2^2 = O(k^{-1}).$$

Hence from (33)

$$\begin{aligned} f_0(\tilde{x}) &\leq f_0(x^*) - \tilde{\lambda} + O(k^{-1} \ln k) + O(k^{-1}) - \frac{1}{2} \|\sqrt{k} g(\tilde{x}) - \frac{v}{\sqrt{k}}\|_2^2 \\ &\leq f_0(x^*) - \tilde{\lambda} + O(k^{-1} \ln k) + O(k^{-1}). \end{aligned} \tag{34}$$

Therefore for $k > 0$ large enough

$$f_0(\tilde{x}) \leq f_0(x^*) - \frac{\tilde{\lambda}}{2}. \tag{35}$$

We show now that for large $k > 0$, $\tilde{x}$ has to be "close" to the feasible region of problem (1). Indeed, from the first inequality in (34) and Assumption A1 we have that

$$\frac{1}{2}\left\|\sqrt{k}g(\tilde{x}) - \frac{v}{\sqrt{k}}\right\|_2^2 \leq f_0(x^*) - \underline{f_0} + O(k^{-1}) + O(k^{-1}\ln k),$$

where $\underline{f_0} = \inf\{f_0(x) \mid x \in L_{f_0(x^*)}(u, v, k_0)\} > -\infty$. Hence $\|\sqrt{k}g(\tilde{x}) - \frac{v}{\sqrt{k}}\|_2^2$ is bounded from above, which implies that

$$\|g(\tilde{x})\|_2^2 \leq O(k^{-1}).$$

Let $\bar{\Omega}_k = \{x : f_i(x) \geq -k^{-1}, i = 1, \ldots, p, |g_j(x)| \leq O(k^{-\frac{1}{2}}), j = 1, \ldots, q\}$. Then $\tilde{x} \in \bar{\Omega}_k$ by the above reasoning and the fact that $\|\cdot\|_2 \geq \|\cdot\|_\infty$. Hence

$$f_0(\tilde{x}) \geq \min\{f_0(x) : x \in \bar{\Omega}_k\}.$$

By the nondegeneracy assumptions C2–C3 and Theorem 6 of [[8], p. 34], which quantifies how a local minimum of problem (1) and its objective value change when that problem is slightly perturbed, and recalling that x^* is a global minimum of problem (1) we obtain

$$f_0(\tilde{x}) \geq f_0(x^*) - k^{-1}\sum_{i=1}^{r} u_i^* - k^{-\frac{1}{2}}\sum_{j=1}^{q} v_j^*.$$

Therefore, for $k \geq k_0$ and $k_0 > 0$ large enough we have

$$f_0(\tilde{x}) \geq f_0(x^*) - \frac{\tilde{\lambda}}{4},$$

which contradicts (35). This completes the proof of the theorem. □

7. Dual problems

While the classical Lagrangian is of great importance in constrained optimization, it has well known drawbacks. First, its unconstrained minimum may fail to exist even for optimal Lagrange multipliers and even when the second-order optimality conditions are fulfilled. Second, the dual function that is based on it is, in general, not smooth even if the functions from which it is formed are smooth.

We show in this section that these basic drawbacks are eliminated by using the MBAL function. Moreover, the dual problem based on this function has some important properties that the dual problem based on the classical Lagrangian lacks. The results below are generalization of results by Rockafellar in [16] for problems with both equality and inequality constraints.

The dual function and the dual problem based on the classical Lagrangian for problem (1) are, respectively,

$$\psi(w) = \psi(u, v) = \inf_{x \in \mathbf{R}^n} L(x, u, v) \tag{36}$$

and

$$w^* = (u^*, v^*) = \operatorname{argmax}\{\psi(u, v) \mid u \in \mathbf{R}_+^p,\ v \in \mathbf{R}^q\}. \tag{37}$$

Consider now the dual function and the dual problem corresponding to the MBAL function $F(x, u, v, k)$:

$$\phi_k(w) = \phi_k(u, v) = \inf_{x \in \mathbf{R}^n} F(x, u, v, k) \tag{38}$$

and

$$w^* = \operatorname{argmax}\{\phi_k(w) \mid w \in \mathbf{R}_+^p \times \mathbf{R}^q\}. \tag{39}$$

Both dual functions $\psi(w)$ and $\phi_k(w)$ are concave and both dual problems (37) and (39) are convex programming problems. However strong duality only holds in general for the "augmented" dual problem (39). Before proving this we first state and prove a result about the continuous differentiability of $\phi_k(w)$.

Lemma 7.1. *If Assumption* A1 *holds, then there exists a* $k_0 > 0$ *and* $\delta > 0$ *such that* $\phi_k(w)$ *is twice continuously differentiable in* $D(w^*, \delta, k_0)$.

Proof: Clearly, $\phi_k(w)$ is concave for every $k > 0$. If $w \in D(w^*, \delta, k_0)$ then by Theorems 5.1 and 5.2, $F(x, u, v, k)$ has a unique minimizer $\hat{x} \equiv \hat{x}(w, k)$ and is strongly convex in a neighborhood of $\hat{x}$. Therefore, $\phi_k(w) = \min_{x \in \mathbf{R}^n} F(x, w, k) = F(\hat{x}(w, k), w, k)$ is smooth and $\nabla_w \phi_k(w) = \nabla_w \hat{x}(\cdot) \nabla_x F(\hat{x}, w, k) + \nabla_w F(\hat{x}, w, k)$. Moreover, for $k \geq k_0$

$$\hat{x}(w^*, k) = x^*.$$

Since $\nabla_x F(x, w, k) = 0$ at $x = \hat{x}(w, k)$ it follows that

$$\nabla_w \phi_k(w) = \nabla_w F(\hat{x}, w, k) = -[k^{-1}(\ln(kf(\hat{x}(\cdot)) + 1)^T, g(\hat{x}(\cdot))]^T.$$

and

$$\nabla_{ww}^2 \phi_k(\cdot) = \nabla_w \hat{x}(\cdot) \nabla_{xw}^2 F(\cdot). \tag{40}$$

By differentiating $\nabla_x F(\hat{x}, w, k) = 0$, with respect to w we obtain

$$\nabla_w \hat{x}(\cdot) = -\nabla_{wx}^2 F(\hat{x}(\cdot), w, k) \cdot \left(\nabla_{xx}^2 F(\hat{x}(\cdot), w, k)\right)^{-1},$$

which when substituted into (40) yields

$$\nabla^2_{ww}\phi_k(\cdot) = -\nabla^2_{wx}F(\cdot)\big(\nabla^2_{xx}F(\hat{x}(\cdot), w, k)\big)^{-1}\nabla^2_{xw}F(\cdot). \tag{41}$$

Note that $\nabla^2_{xx}F(\cdot)^{-1}$ exists because $F(x, w, k)$ is strongly convex in a neighborhood of $\hat{x}$. Finally,

$$\nabla^2_{xw}F(\cdot) = \nabla^2_{wx}F(\cdot)^T = -[\nabla f(\hat{x}(\cdot))^T\hat{D},\ \nabla g(\hat{x}(\cdot))^T]. \tag{42}$$

□

It is well known that for nonconvex optimization the basic duality theorem of convex programming is not true. However, for the dual problem based on the MBAL function, if the second-order sufficient conditions hold, then the basic duality theorem remains true and the second-order sufficient conditions are satisfied for the dual problem. The following theorem, which is an analog of Theorem 4 in [12], is a statement of these facts.

Theorem 7.2. *Under the second-order sufficient conditions* C1–C3 *and Assumption* A1, *there exists* $k_0 > 0$, *such that for any* $k \geq k_0$ *the following statements are true*:

(i) *The existence of a solution* x^* *to the primal problem* (1) *guarantees that the dual problem* (39) *has a solution* $w^* = (u^*, v^*)$ *and that*

$$f_0(x^*) = \phi_k(u^*, v^*).$$

(ii) *The second-order sufficient conditions are satisfied for the dual problem* (39).

(iii) $(x^*, w^*) = (x^*, u^*, v^*)$ *is a solution to the primal and dual problems* (1) *and* (39) *if and only if it is a saddle point of* $F(x, w, k)$; *i.e.*,

$$F(x, w^*, k) \geq F(x^*, w^*, k) \geq F(x^*, w, k), \quad \forall x \in \mathbf{R}^n, \quad w \in \mathbf{R}^p_+ \times \mathbf{R}^q.$$

Proof: Our proof of parts (i) and (ii) are similar to the proof of Theorem 3 in [12].

(i) Due to C1 there exists a $w^* = (u^*, v^*) \in \mathbf{R}^p_+ \times \mathbf{R}^q$ such that the Karush-Kuhn-Tucker conditions hold. Let us prove that w^* is the solution of the dual problem (39). First, note that (39) is a convex programming problem since dual function $\phi_k(w)$ is concave. Also $\phi_k(w)$ is smooth and

$$\nabla_w\phi_k(w^*)$$
$$= -[\overbrace{0, \ldots, 0}^{r}, k^{-1}\ln(kf_{r+1}(x^*) + 1), \ldots, k^{-1}\ln(kf_p(x^*) + 1), \overbrace{0, \ldots, 0}^{q}],$$

or

$$\begin{aligned} u_i^* > 0 &\Rightarrow \partial_{u_i}\phi_k(w^*) = 0, \quad i = 1, \ldots, r,\\ u_i^* = 0 &\Rightarrow \partial_{u_i}\phi_k(w^*) < 0, \quad i = r+1, \ldots, p,\\ &\quad\ \partial_{v_j}\phi_k(w^*) = 0, \quad j = 1, \ldots, q. \end{aligned}$$

In other words, the optimality conditions for the convex program (39) are satisfied at $w = w^* = (u^*, v^*)$, therefore $\phi_k(w^*) = \max\{\phi_k(w) \mid w \in \mathbf{R}^p_+ \times \mathbf{R}^q\}$, and

$$\phi_k(w^*) = \inf_x F(x, w^*, k) = F(x^*, w^*, k) = f_0(x^*).$$

(ii) Consider the classical Lagrangian

$$\mathcal{L}(w, \lambda, k) = \phi_k(w) - \sum_{i=1}^{p} \lambda_i u_i$$

for the dual problem (39). Then

$$\nabla^2_{ww}\mathcal{L}(w, \lambda, k) = \nabla^2_{ww}\phi_k(w) \tag{43}$$

and the affine subspace tangent to the set of feasible dual solutions at $w^* = (u^*, v^*)$ is $Y = \{y \in \mathbf{R}^p_+ \times \mathbf{R}^q : (y_{r+1} = y_{r+2} = \ldots = y_p = 0)\}$. Using condition C1 and formulas (41), (42) and (43) and by employing considerations similar to those, which has been used to prove Theorem 3 in [12] one can show that for $k \geq k_0$ and k_0 large enough,

$$y^T \nabla^2_{ww}\mathcal{L}(w^*, \lambda^*, k) y < 0, \quad \forall y \in Y.$$

The gradients $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, $i = r+1, \ldots, p$ of the active constraints for the dual problem are linearly independent and corresponding Lagrange multipliers $\lambda_i^* = k^{-1}\ln(kf_i(x^*) + 1) > 0$, $i = r+1, \ldots, p$. Consequently, this proves that the second-order sufficient conditions are satisfied for the dual problem (39).

(iii) Suppose that (x^*, w^*) is a solution to the primal-dual pair of problems (1) and (39). Then from Theorem 5.2 it immediately follows that

$$F(x, w^*, k) \geq F(x^*, w^*, k) \quad \forall x \in \mathbf{R}^n.$$

Now $g_i(x^*) = 0$, $i = 1, \ldots, q$, and $\ln(kf_i(x^*) + 1) = 0$, $i = 1, \ldots, r$. Therefore, replacing v^* by any $v \in \mathbf{R}^q$ and $u^*_{(r)}$ by any $u_{(r)} \in \mathbf{R}^r_+$ does not change $F(x^*, w^*, k)$. Also, since $\ln(kf_i(x^*) + 1) > 0$, $i = r+1, \ldots, p$, $F(x^*, w^*, k)$ does not increase if $u^*_{(p-r)} = 0$ is replaced by any $u_{(p-r)} \in \mathbf{R}^{p-r}_+$. Thus we have that

$$F(x^*, w^*, k) \geq F(x^*, w, k) \quad \forall w \in \mathbf{R}^p_+ \times \mathbf{R}^q,$$

which concludes our proof that the primal-dual solution (x^*, w^*) is a saddle point of $F(x, w, k)$.

Now suppose (x^*, w^*) is a saddle point of $F(x, w, k)$ and $f_i(x^*) < 0$ for some i. Let $\bar{u} = (u_1^*, \ldots, u_{i-1}^*, u_i^* + \epsilon, u_{i+1}^*, \ldots, u_p^*)$, for some $\epsilon > 0$, and $\bar{v} = v^*$. Then clearly, $F(x^*, w^*, k) < F(x^*, \bar{w}, k)$, which contradicts our assumption that (x^*, w^*)

is a saddle point. Assume now that for some i, $g_i(x^*) \neq 0$. Let $\bar{u} = u^*$ and $\bar{v} = v^*$ except that $\bar{v}_i = v_i^* + \epsilon$ if $g_i(x^*) > 0$ and $\bar{v}_i = v_i^* - \epsilon$ if $g_i(x^*) < 0$. Then again $F(x^*, w^*, k) < F(x^*, \bar{w}, k)$, and we have a contradiction. Hence, we conclude that x^* is feasible for the primal problem.

Assume that for some i, $f_i(x^*) > 0$ and $u_i^* > 0$, and let $\bar{u} = (u_1^*, \ldots, u_{i-1}^*, u_i^* - \epsilon, u_{i+1}^*, \ldots, u_p^*)$, for $u_i^* > \epsilon > 0$, and $\bar{v} = v^*$. Again $F(x^*, w^*, k) < F(x^*, \bar{w}, k)$, which contradicts our assumption that (x^*, w^*) is a saddle point. $u_i^* f_i(x^*) = 0$, $i = 1, \ldots, p$ and it follows from the dual feasibility of w^* that for any x feasible for the primal problem,

$$\begin{aligned} f_0(x) &\geq f_0(x) - k^{-1}\sum_{i=1}^{p} u_i^* \ln(kf_i(x) + 1) - \sum_{j=1}^{q} v_j^* g_j(x) + k/2\sum_{j=1}^{q} g_j^2(x) \\ &= F(x, w^*, k) \geq F(x^*, w^*, k) = f_0(x^*). \end{aligned}$$

Hence x^* solves the primal problem (1) and w^* solves the dual problem (39). □

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