Book Orders for Market Scoring Rules

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Abstract

This explains how to smoothly integrate booked orders with a combinatorial market maker, all for the general case of bets on $E[x|A]$ for arbitrary random variables x and sets A.

Let there be a complete set I of possible states i. To bet on the probability $P(i)$, one can buy an asset of the form "Pays \$1 if true state is i" for a cash price \$ p_i. This looks like a good deal if one believes $P(i) > p_i$. To bet on the probability $P(E)$ of an set E of mutually exclusive possible states i, one can buy an asset of the form "Pays \$1 if true state $i \in E$" for a cash price $p_E = \sum_{i \in E} p_i$. To bet on the expected value of any random variable x_i over these states, once can buy an asset of the form "Pays \$ $x_{\text{true } i}$," paying a cash amount $\mathrm{E}_p[x] = \sum_{i \in I} p_i x_i$. (Without loss of generality we can consider only variables where $x_i \in [0, 1]$.) To bet on the conditional probability $P(E|A)$, one can buy an asset of the form "Pays \$1 if $E \cap A$" for the price of $p_{E|A}$ units of "Pays \$1 if A," where $p_{E|A} = p_{E \cap A}/p_A$.

In general, one can bet on $\mathrm{E}_p[x|A]$ by buying an asset of the form "Pays \$ x_i if A," in trade paying $\hat{x}$ units of assets of the form "Pays \$1 if A," where

$$\hat{x} = \mathrm{E}_p[x|A] = \frac{\sum_{i \in A} p_i x_i}{\sum_{i \in A} p_i}.$$

This looks like a good deal when one believes that $\mathrm{E}_P[x|A] > \mathrm{E}_p[x|A]$. If one instead believes that $\mathrm{E}_P[x|A] < \mathrm{E}_p[x|A]$, one can sell this bet instead of buying, which is equivalent to buying a bet on the random variable $y_i = 1 - x_i$. We can thus without loss of generality consider only offers to buy. The above descriptions can all be considered to define one quantity unit $q = 1$ of that sort of trade.

A logarithmic market scoring rule (or combinatorial market maker) maintains prices p_i on some complete set of states i. These prices are open to all, allowing anyone to make

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any bets expressable in terms of random variables defined on these states. These prices are only valid for infinitessimal quantities dq_i, however, and change with the net quantities q_i purchased so far. If net quantities are initially zero, and prices are initially p_{0i}, then for any resulting set of quantities q_i, the resulting prices are

$$p_i = \frac{p_{0i} \exp(q_i/b)}{\sum_j p_{0j} \exp(q_j/b)},$$

where b is a parameter describing the over all level of subsidy that the market maker provides to traders. (Starting from uniform prices, the maximum loss is $b \log(|I|)$.) One can equilvanently say that when the current prices are p_{0i}, then one can change those prices to p_i if one acquires assets $q_i = b \log(p_i/p_{0i})$. (Acquiring negative assets means you have to pay that many of them.)

A user with sufficient assets could make an offer to buy up to q' units of bets on $\mathrm{E}[x|A]$, for a maximum marginal price of $\hat{x}'$, which expires on some date. If the current market maker prices were then p_{0i}, a quantity q bought of bets on $\mathrm{E}_P[x|A]$ would produce a new market estimate $\mathrm{E}_p[x|A]$ of

$$\hat{x}(q) = \frac{\sum_{i \in A} x_i p_{0i} \exp(q x_i/b)}{\sum_{i \in A} p_{0i} \exp(q x_i/b)}.$$

Thus if $\hat{x}(q') \leq \hat{x}'$, the entire new order could be used up trading with the market maker. If instead $\hat{x}(q') > \hat{x}'$, a portion $\tilde{q} = \max(0, \hat{x}^{-1}(\hat{x}'))$ could be traded with the market maker, leaving a book order for the remaining quantity $q' - \tilde{q}$.

If new orders get the advantage of any order price overlap, then each book order with a limit $\hat{x}'$ imposes a constraint on the market maker prices p_i. It says that until that order expires, or all its quantity is used, the prices must satisfy $\hat{x} \leq \hat{x}'$. This constraint is binding on a particular plane in the price vector space. Thus the above description of a new order trading with the market maker is only valid until the price vector reaches one of these planes. At that point the new order trades with both the market maker and the book order at the same time, moving the market maker prices in the plane of the book order. This continues until a price or quantity limit of one of the orders is reached, or until another book order plane is reached.

Let a new order on $\mathrm{E}[x|A]$ with limits q', $\hat{x}'$ trade with both the market maker, starting at prices p_{0i}, and with a single book order on $E[y|B]$ with limits r', $\hat{y}'$. If the new order traded a quantity q and the book order traded a quantity r, the new expected values would be

$$\begin{aligned}
\hat{x}(q,r) &= \frac{\sum_{i \in A} x_i p_{0i} \exp((q x_i + r y_i \delta_{iB})/b)}{\sum_{i \in A} p_{0i} \exp((q x_i + r y_i \delta_{iB})/b)} \\
\hat{y}(q,r) &= \frac{\sum_{i \in B} y_i p_{0i} \exp((q x_i \delta_{iA} + r y_i)/b)}{\sum_{i \in B} p_{0i} \exp((q x_i \delta_{iA} + r y_i)/b)}
\end{aligned}$$

where δ_{iE} is 1 when $i \in E$ and 0 otherwise. These two functions $\hat{x}(q,r)$ and $\hat{y}(q,r)$ are all that one needs to calculate the largest trade q possible while preserving the constraints $q \in [0, q']$, $r \in [0, r']$, $\hat{y}(q,r) = \hat{y}'$, $\hat{x}(q,r) \leq \hat{x}'$, and the similar expected value inequalities for other books.

More generally, let a single new order trade with N book orders, starting from prices p_{0i} where all these book order price constraints are binding. Each order bets on $\mathrm{E}[x_n|A_n]$ with limits q'_n, $\hat{x}'_n$, where $n = 0$ identifies the new order and $n \in [1, N]$ are the book orders. If we let $\vec{q} = \{q_n\}_n$, we can write

$$\hat{x}_n(\vec{q}) = \frac{\sum_{i \in A_n} x_{ni} p_{0i} \exp(\sum_{m=0}^{N} q_m x_{mi} \delta_{iA_m}/b)}{\sum_{i \in A_n} p_{0i} \exp(\sum_{m=0}^{N} q_m x_{mi} \delta_{iA_m}/b)}.$$

Initially $\hat{x}_n = \hat{x}_n(\vec{0})$, and then these functions $\hat{x}_n(\vec{q})$ are all one needs to calculate the largest trade q_0 possible while preserving the constraints $q_n \in [0, q'_n]$ for $n \in [0, N]$, $\hat{x}_n(\vec{q}) = \hat{x}'_n$ for $n \in [1, N]$, $\hat{x}_0(\vec{q}) \leq \hat{x}'_0$, and constraints like $\hat{x}_n(\vec{q}) \leq \hat{x}'_n$ for all the other book orders. Once one of these constraints becomes binding, the new order may continue trading with a new set of book orders.

Iterative numerical methods will probably be required to make each such computation. The computational cost of this may be reducable by grouping the states i into sets where all of the $n \in [0, N]$ have the same values of x_{in}, δ_{iA_n}.