Quantum Probability From Decision Theory and Exchangeability

Robin Hanson *
Department of Economics
George Mason University†

July 27, 2001

Abstract

Deutsch's derivation of quantum probability from decision theory has been criticized as relying on an implausible hidden assumption. I here offer as a substitute an apparently more plausible exchangeability assumption.

Introduction

Many have attempted to derive the standard Born rule for quantum probabilities from other well-accepted or a priori plausible considerations. These attempts have been widely criticized, however, for consistently making implicit but crucial assumptions for which no justification is offered (Kent, 1990). David Deutsch has recently offered a derivation based on some basic decision theory considerations (Deutsch, 1999). While this has induced some sympathetic response (Forster, 1999; Gill, 2001), Deutsch's attempt has also been criticized by a group of five scholars as making implicit unjustified assumptions (Barnum, Caves, Finkelstein, Fuchs, & Schack, 2000; Finkelstein, 1999).

This note attempts to reconcile Deutsch's approach with the concerns of his critics. It does this by accepting those critic's more precise formulation of Deutsch's derivation, and within that more precise formulation offering and defending a new assumption sufficient to reach Deutsch's conclusion. This will hopefully help the discussion to move from the details of Deutsch's derivation to the plausibility of the assumptions required.

*For their comments, I thank David Deutsch and Jerry Finkelstein.

† *rhanson@gmu.edu* http://hanson.gmu.edu 704-993-2326 FAX: 704-993-2323 MSN 1D3, Carow Hall, Fairfax VA 22030

Analysis

Following the notation of Deutsch's critics (Barnum et al., 2000), I assume a rational decision maker who makes choices in order to maximize a value function $\mathcal{V}(|\psi\rangle;\hat{X})$, where $|\psi\rangle$ is a quantum state and $\hat{X}$ is a Hermitian utility operator. That is, a decision maker characterized by $\hat{X}$ will choose $|\psi\rangle$ over $|\varphi\rangle$ if $\mathcal{V}(|\psi\rangle;\hat{X}) > \mathcal{V}(|\varphi\rangle;\hat{X})$. Like any Hermitian operator, $\hat{X}$ can be expressed in terms of an orthonormal set of eigenvectors $|\phi_j\rangle$ and associated eigenvalues x_j, as

$$\hat{X} = \sum_j x_j \hat{\Pi}_j,$$

where $\hat{\Pi}_j = |\phi_j\rangle\langle\phi_j|$ are basis operators.

Together with Deutsch and his critics, let us assume that $\mathcal{V}(|\psi\rangle;\hat{X})$ satisfies the following two conditions:

$$\mathcal{V}(|\psi\rangle;\sum_j(x_j+k)\hat{\Pi}_j) = k + \mathcal{V}(|\psi\rangle;\sum_j x_j\hat{\Pi}_j) \tag{1}$$

$$\mathcal{V}(|\psi\rangle;\sum_j(-x_j)\hat{\Pi}_j) = -\mathcal{V}(|\psi\rangle;\sum_j x_j\hat{\Pi}_j) \tag{2}$$

The first equation can be thought of as a *sure thing* principle. Gaining k utility in every utility basis state is the same as gaining k utility overall. The second equation can be thought of as a *no free lunch* principle. Losing a schedule of utilities per states is as bad as it is good to gain those same utilities per states. That is, one should be indifferent between a certain utility for sure and a fifty/fifty lottery over possibly gaining or losing any schedule of utility changes per basis state.

Let us now consider what Deutsch and his critics all consider to be the "pivotal" result of his paper. For this result, we restrict attention to the case of two basis states, so that

$$|\psi\rangle = \lambda_1|\phi_1\rangle + \lambda_2|\phi_2\rangle.$$

The pivotal result is that

$$\mathcal{V}(\frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle);x\hat{\Pi}_1 + y\hat{\Pi}_2) = \frac{1}{2}(x+y), \tag{3}$$

That is, decisions about states with identical mixing coefficients should be treated like decisions when there are equal probabilities of achieving the two states.

To derive this result, we can substitute $k = -x - y$ into equation 1 to obtain

$$\mathcal{V}(|\psi\rangle;x\hat{\Pi}_1 + y\hat{\Pi}_2) - x - y = \mathcal{V}(|\psi\rangle;-y\hat{\Pi}_1 - x\hat{\Pi}_2)$$

Using equation 2, we can transform the right hand side, and get

$$\mathcal{V}(|\,\psi\,\rangle; x\hat{\Pi}_1 + y\hat{\Pi}_2) - x - y = -\mathcal{V}(|\,\psi\,\rangle; y\hat{\Pi}_1 + x\hat{\Pi}_2).$$

The pivotal result of equation 3 could be easily obtained from this if only we knew that

$$\mathcal{V}(\frac{1}{\sqrt{2}}(|\,\phi_1\,\rangle + |\,\phi_2\,\rangle); x\hat{\Pi}_1 + y\hat{\Pi}_2) = \mathcal{V}(\frac{1}{\sqrt{2}}(|\,\phi_2\,\rangle + |\,\phi_1\,\rangle); x\hat{\Pi}_2 + y\hat{\Pi}_1). \tag{4}$$

This is the assumption which Deutsch's critics call "hidden" because Deutsch's notation did not make it as easy to see.

Deutsch's critics claim that equation 4 is as "equally well (or badly) justified" as the stronger assumption that

$$\mathcal{V}(|\,\phi\,\rangle; x\hat{\Pi}_1 + y\hat{\Pi}_2) = \mathcal{V}(|\,\phi\,\rangle; y\hat{\Pi}_1 + x\hat{\Pi}_2),$$

which implies that *all* decisions about mixtures of two states should treat them like fifty/fifty lotteries over those states.

I want to argue that a more reasonable generalization of the required assumption of equation 4 is

$$\mathcal{V}(\lambda_x|\,\phi_1\,\rangle + \lambda_y|\,\phi_2\,\rangle; x\hat{\Pi}_1 + y\hat{\Pi}_2) = \mathcal{V}(\lambda_y|\,\phi_1\,\rangle + \lambda_x|\,\phi_2\,\rangle; y\hat{\Pi}_1 + x\hat{\Pi}_2). \tag{5}$$

This *exchangeability* assumption says that all that should matter for decisions is the set of mixing coefficient and utility pairings, (λ_j, x_j), across the basis states; it shouldn't matter which particular physical basis states embodies which pairing.

For example, imagine that a polarizing filter stands between you and deadly x-rays. If the polarizer is oriented vertically, a horizontally-polarized photon will kill you, but a vertically-polarized photon will leave you unharmed. In this context, you might need to consider your values regarding various possible photon polarization states. The symmetry assumption says that you should have exactly the same values regarding a situation with a horizontally-oriented polarizer, and considering "axis-reversed" photons, i.e., where the old vertical mixing coefficient is now the horizontal mixing coefficient and vice-versa. (For circularly polarized photons the new photons have opposite circular polarization plus a phase shift). It doesn't fundamentally matter to you how photons are polarized; what matters to you is the *mapping* between polarization states and whether you live or die.

Conclusion

I have proposed an exchangeability assumption to be used in Deutsch's derivation of the Born rule for quantum probabilities. I consider this assumption to be what Deutsch had in mind in his original presentation. Some may, with me, consider this assumption to be more a prior plausible than the stronger assumption, that Deutsch's critics claim is equally justified, which implies that one should treat all mixtures the same.

Of course one need not accept Deutsch's derivation, even if one accepts an exchangeability assumption. The sure-thing assumption seems hard to question, but to some the no-free-lunch assumption may seem less obviously compelling.

References

Barnum, H., Caves, C. M., Finkelstein, J., Fuchs, C. A., & Schack, R. (2000). Quantum Probability from Decision Theory?. *Proceedings of the Royal Society of London*, *A456*, 1175–1182.

Deutsch, D. (1999). Quantum Theory of Probability and Decisions. *Proceedings of the Royal Society of London*, *A455*, 3129–3197.

Finkelstein, J. (1999). Quantum Probability from Decision Theory?. http://xxx.lanl.gov/abs/quant-ph/9907004.

Forster, M. R. (1999). A Note on Deutsch's Quantum Mechanics and Decision. http://philosophy.wisc.edu/forster/papers/Deutsch.htm.

Gill, R. D. (2001). On an Argument of David Deutsch. http://www.math.uu.nl/people/gill/Preprints/deutsch.pdf.

Kent, A. (1990). Against Many-Worlds Interpretations. *International Journal of Modern Physics A*, *5*, 1745–1762.