20.7 Inventory Model

In Chap. 15 the following inventory problem was considered. A camera store stocks a particular model camera that can be ordered weekly. Let $D_1, D_2, \ldots$, represent the demand for this camera during the first week, the second week, . . . , respectively. It is assumed that the D_t are independent, identically distributed random variables having a Poisson distribution with parameter λ equal to 1. Let X_0 represent the number of cameras on hand at the outset, X_1 the number of cameras on hand at the end of week one, X_2 the number of cameras on hand at the end of week two, and so forth. On Saturday night the store places an order that is delivered in time for the opening of the store on Monday. The store uses an (s, S) ordering policy. If the number of cameras on hand at the end of the week is less than $s = 1$ (no cameras in stock), the store orders up to $S = 3$. Otherwise, the store does not order (if there are any cameras in stock, no order is placed). It is assumed that sales are lost when demand exceeds the inventory on hand (no backlogging). The cost structure considered calls for incurring a penalty cost of \$50 per unit for each unit of unsatisfied demand (lost sales). If $z > 0$ cameras are ordered, the cost incurred is $10 + 25z$ dollars. If no cameras are ordered, no ordering cost is incurred. Holding costs are to be neglected. In Sec. 15.7, this policy was evaluated by using the (long-run) expected average cost per unit time as the criterion. It is not evident that this policy is optimal, and the purpose of this section is to find the optimal policy. Even though we know that the optimal policy must be of the (s, S) form, we shall consider all possible policies, although we shall assume that three cameras is the maximum number of cameras that the store will stock. The *policy improvement algorithm* will be used first, followed by the *linear programming formulation*.

Because X_t represents the state of the system, i.e., the number of cameras on hand at the end of week t (before ordering), then $X_t = 0, 1, 2, 3$. Similarly, there are four possible decisions:

Decision	Action
0	Do not order
1	Order 1 camera
2	Order 2 cameras
3	Order 3 cameras

The possible transitions are given by[1]

	Decision 0			
State	0	1	2	3
0	1	0	0	0
1	$P\{D \geq 1\}$	$P\{D = 0\}$	0	0
2	$P\{D \geq 2\}$	$P\{D = 1\}$	$P\{D = 0\}$	0
3	$P\{D \geq 3\}$	$P\{D = 2\}$	$P\{D = 1\}$	$P\{D = 0\}$

[1] Note that in this example the set of possible decisions varies with the states.

State	Decision 1: 0	1	2	3
0	$P\{D \geq 1\}$	$P\{D = 0\}$	0	0
1	$P\{D \geq 2\}$	$P\{D = 1\}$	$P\{D = 0\}$	0
2	$P\{D \geq 3\}$	$P\{D = 2\}$	$P\{D = 1\}$	$P\{D = 0\}$
3	Decision 1 not permitted			

State	Decision 2: 0	1	2	3
0	$P\{D \geq 2\}$	$P\{D = 1\}$	$P\{D = 0\}$	0
1	$P\{D \geq 3\}$	$P\{D = 2\}$	$P\{D = 1\}$	$P\{D = 0\}$
2, 3	Decision 2 not permitted			

State	Decision 3: 0	1	2	3
0	$P\{D \geq 3\}$	$P\{D = 2\}$	$P\{D = 1\}$	$P\{D = 0\}$
1, 2, 3	Decision 3 not permitted			

Recalling that the demand D is a Poisson random variable with parameter $\lambda = 1$, and using appendix Table A.5.4, these transitions can now be expressed as

State	Decision 0: 0	1	2	3
0	1	0	0	0
1	0.632	0.368	0	0
2	0.264	0.368	0.368	0
3	0.080	0.184	0.368	0.368

State	Decision 1: 0	1	2	3
0	0.632	0.368	0	0
1	0.264	0.368	0.368	0
2	0.080	0.184	0.368	0.368
3	Decision 1 not permitted			

State	Decision 2: 0	1	2	3
0	0.264	0.368	0.368	0
1	0.080	0.184	0.368	0.368
2, 3	Decision 2 not permitted			

State	Decision 3: 0	1	2	3
0	0.080	0.184	0.368	0.368
1, 2, 3	Decision 3 not permitted			

The cost information required is similar to that given in Sec. 15.7, and you are urged to review this material. A summary is given by

State	Decision	Actual Cost Per Week	Expected Cost Per Week, C_{ik}
0	0	$50D$	$50E(D) = 50$
	1	$35 + 50 \max\{(D-1), 0\}$	$35 + 50[1P\{D=2\} + 2P\{D=3\} + \cdots] = 53.4$
	2	$60 + 50 \max\{(D-2), 0\}$	$60 + 50[1P\{D=3\} + 2P\{D=4\} + \cdots] = 65.2$
	3	$85 + 50 \max\{(D-3), 0\}$	$85 + 50[1P\{D=4\} + 2P\{D=5\} + \cdots] = 86.2$
1	0	$50 \max\{(D-1), 0\}$	$50[1P\{D=2\} + 2P\{D=3\} + \cdots] = 18.4$
	1	$35 + 50 \max\{(D-2), 0\}$	$35 + 50[1P\{D=3\} + 2P\{D=4\} + \cdots] = 40.2$
	2	$60 + 50 \max\{(D-3), 0\}$	$60 + 50[1P\{D=4\} + 2P\{D=5\} + \cdots] = 61.2$
	3	Decision 3 not permitted	
2	0	$50 \max\{(D-2), 0\}$	$50[1P\{D=3\} + 2P\{D=4\} + \cdots] = 5.2$
	1	$35 + 50 \max\{(D-3), 0\}$	$35 + 50[1P\{D=4\} + 2P\{D=5\} + \cdots] = 36.2$
	2, 3	Decisions 2, 3 not permitted	
3	0	$50 \max\{(D-3), 0\}$	$50[1P\{D=4\} + 2P\{D=5\} + \cdots] = 1.2$
	1, 2, 3	Decisions 1, 2, 3 not permitted	

Choose the (s, S) policy already introduced as the initial policy for carrying out the value-determination step (step 1) of the *policy improvement algorithm*. This policy, R_1, calls for ordering up to 3 units whenever the system is in state 0 (no cameras on hand); otherwise, no order is placed. With this policy, the following four equations must be solved simultaneously for $g(R_1)$, $v_0(R_1)$, $v_1(R_1)$, and $v_2(R_1)$ [recall that $v_3(R_1)$ is arbitrarily taken to be zero]:

$$
\begin{aligned}
g(R_1) &= C_{0k_1} + \sum_{j=0}^{3} p_{0j}(k_1)v_j(R_1) - v_0(R_1) \\
&= C_{1k_1} + \sum_{j=0}^{3} p_{1j}(k_1)v_j(R_1) - v_1(R_1) \\
&= C_{2k_1} + \sum_{j=0}^{3} p_{2j}(k_1)v_j(R_1) - v_2(R_1) \\
&= C_{3k_1} + \sum_{j=0}^{3} p_{3j}(k_1)v_j(R_1) - v_3(R_1),
\end{aligned}
$$

or, alternatively,

$$
\begin{aligned}
g(R_1) &= 86.2 + 0.080v_0(R_1) + 0.184v_1(R_1) + 0.368v_2(R_1) - v_0(R_1) \\
&= 18.4 + 0.632v_0(R_1) + 0.368v_1(R_1) - v_1(R_1) \\
&= 5.2 + 0.264v_0(R_1) + 0.368v_1(R_1) + 0.368v_2(R_1) - v_2(R_1) \\
&= 1.2 + 0.080v_0(R_1) + 0.184v_1(R_1) + 0.368v_2(R_1).
\end{aligned}
$$