

Basic Physics

Chapter 3

Rigid Bodies

$x \rightarrow$  Displacement (Linear)

$v = \dot{x} \rightarrow$  linear velocity

$\ddot{x} \rightarrow$  linear acceleration

$\ddot{x} = a$

Rotation

$\theta \rightarrow$  angular displacement

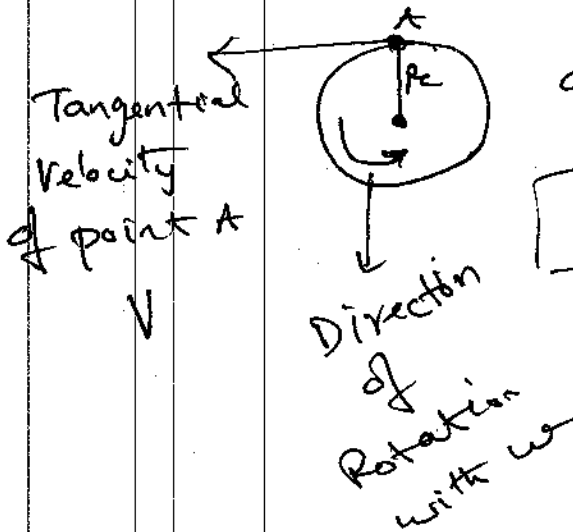
$\omega = \dot{\theta} \rightarrow$  angular velocity

$\ddot{\theta} \rightarrow$  angular acceleration

$\ddot{\theta} = \alpha$

$\downarrow$   
alpha

$r \rightarrow$  radius.



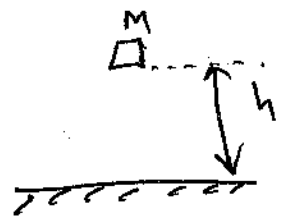
angular velocity =  $\omega$

$$V = r \omega$$

$$\dot{V} = r \dot{\omega}$$

$$a = r \alpha$$

Potential Energy  $PE = mgh$



Linear Kinetic Energy  $KE = \frac{1}{2} m v^2$

Rotational  $KE = \frac{1}{2} I \omega^2$

$I \rightarrow$  moment of Inertia

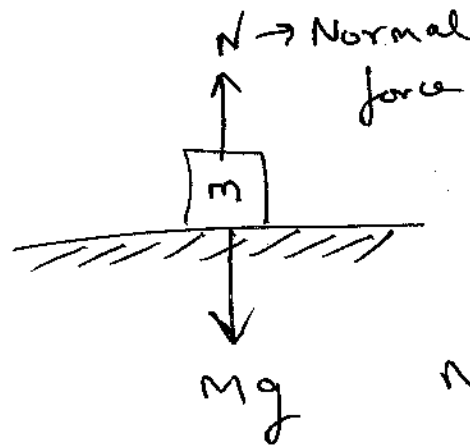
Torque =  $I\alpha$  → Rotating (motion)

Force =  $ma$  → Linear / Translation (motion)

Solve in Terms of Torque or force depending on the type of motion.

Sometimes it is both

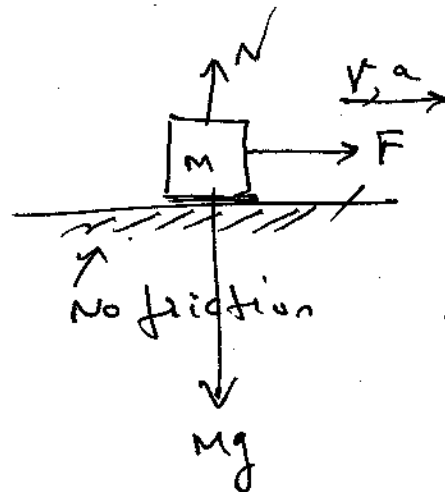
Static



What are the forces on  $M$

$$N = mg$$

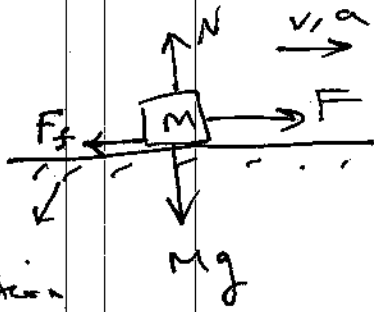
Dynamic



$$N = mg$$

$$F = Ma$$

Force contributes toward  $M$ 's acceleration (Newton's 2<sup>nd</sup> law)



$F_f \rightarrow$  Frictional force due to coefficient friction  $\mu$

(Opposes motion)

Friction  $\mu$

$$N = Mg$$

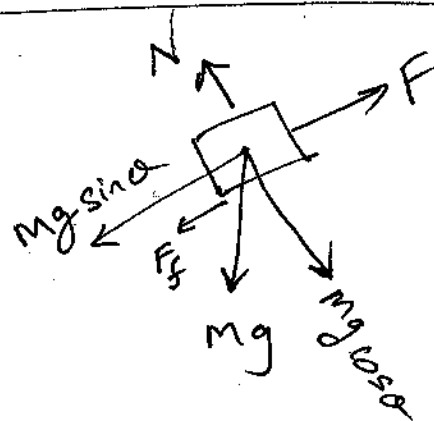
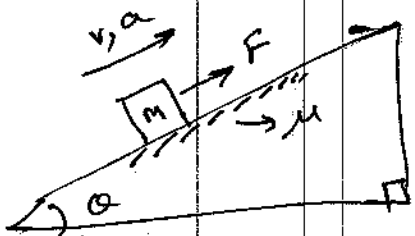
$F > F_f$  (F has to overcome frictional force  $F_f$ )

$$F - F_f = ma$$

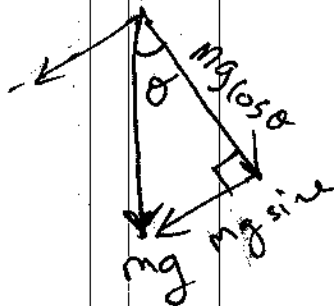
Resultant force

$$\text{If } F \leq F_f$$

No movement at all  
 $v = 0$   
 $a = 0$



Resolve  $Mg$  along the plane & perpendicular to the plane.



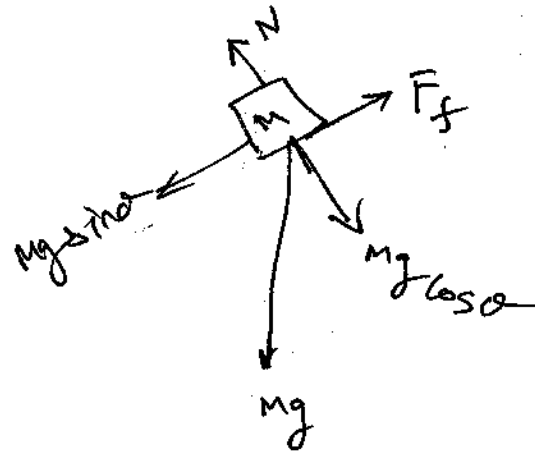
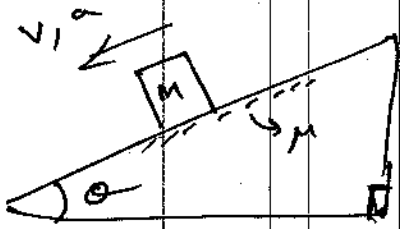
$$N = Mg \cos \alpha$$

$$F > Mg \sin \alpha + F_f$$

$$F - Mg \sin \alpha - F_f = ma$$

$$F_f = \mu N$$

m is sliding down the slope. (4)

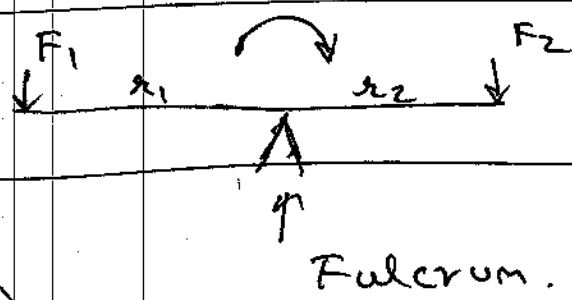


$$N = mg \cos \alpha$$

$$F_f = \mu N$$

$$mg \sin \alpha > F_f$$

$$mg \sin \alpha - F_f = ma$$



Rotation

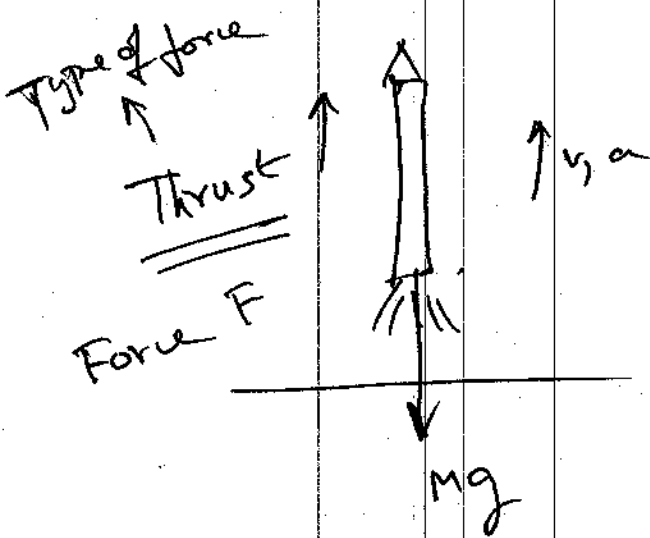
Solve in terms of Torque and NOT IN terms of force

DO NOT SAY  $F_2 > F_1$  INCORRECT

$$F_2 r_2 > F_1 r_1$$

Torque<sub>2</sub> > Torque<sub>1</sub>

$$F_2 r_2 - F_1 r_1 = I \alpha$$

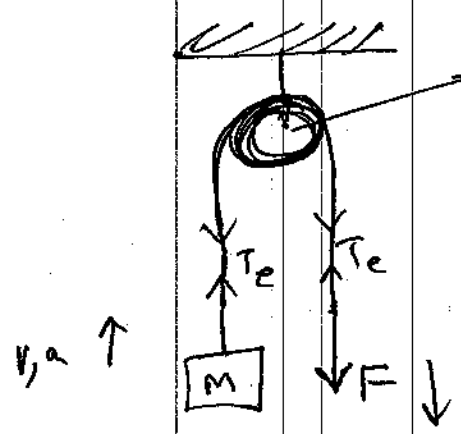


$$F > Mg$$

~~F = Ma~~  
~~F = Ma~~

$$F - Mg = ma$$

$$F - Mg = M \ddot{x}$$



Pulley radius  $r$   
 $m_p$  ~~is~~ Mass = 0  $\rightarrow$  mass of pulley  
 Write the equation of motion.

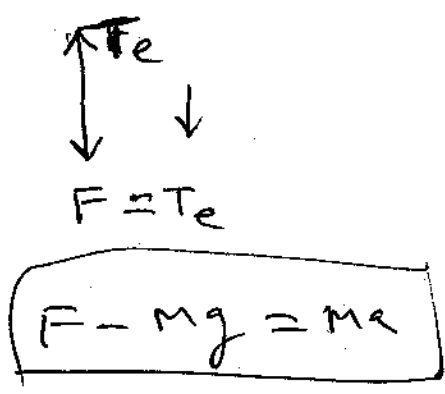
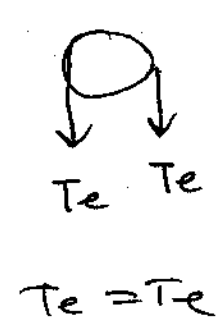
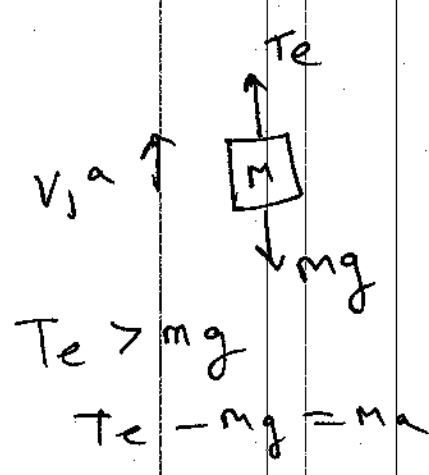
$M = 0$  means that there is no contribution of energy by the

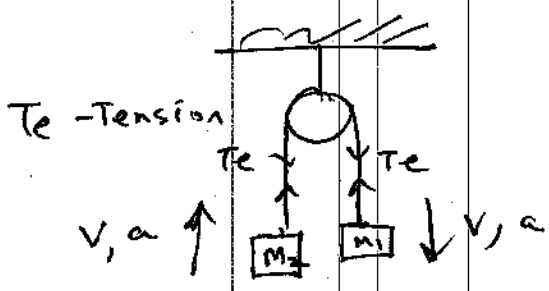
pulley.  $I = 0$

$\uparrow$   
 moment of inertia.

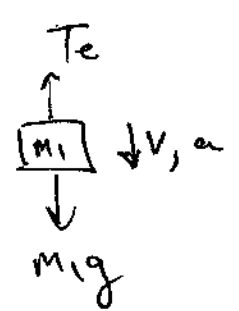
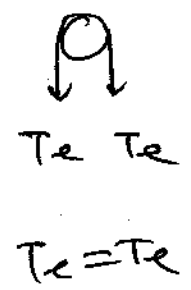
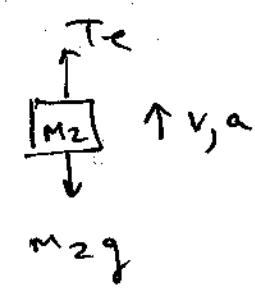
Pulley adds zero torque to the system.

Tension  $T_e$   
Force type  
on strings





$M_p - \text{pulley mass} = 0$   
 $I = 0$



$T_e > m_2 g$

$T_e - m_2 g = m_2 a$

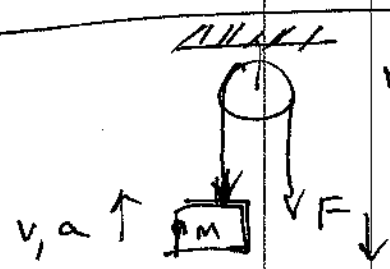
$T_e = m_2 a + m_2 g$

$m_1 g > T_e$

$m_1 g - T_e = m_1 a$

$m_1 g - (m_2 a + m_2 g) = m_1 a$

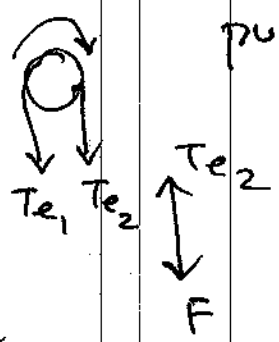
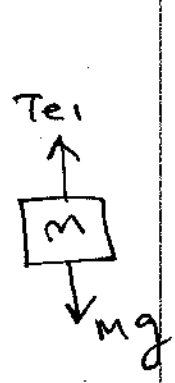
$(m_1 - m_2) g = (m_1 + m_2) a$



$M_p \neq 0$  pulley has mass  $m_p$   
 $I \rightarrow$  moment of ~~the~~ inertia  
 $r \rightarrow$  radius of the pulley

Cannot ignore the pulley

pulley contributes energy to the system.



$T_{e1} \neq T_{e2}$

Tensions are not equal on either side of the pulley

$$T_{e1} > mg$$

$$T_{e1} - mg = ma$$

$$\frac{Fr - Id}{r} - mg = ma$$

$$F - \frac{Id}{r} - mg = ma$$

$$v = r\omega$$

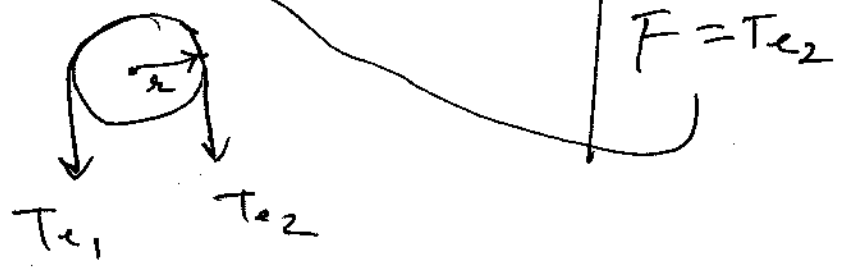
$$a = r\alpha$$

$$F - \frac{Ia}{r^2} - mg = ma$$

$$F - mg = \left( m + \frac{I}{r^2} \right) a$$

$$T_{e2} > T_{e1} \text{ (Force)}$$

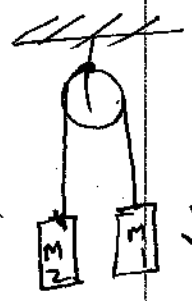
$$rT_{e2} - T_{e1}r = Id \text{ (Torque)}$$



$$Fr - T_{e1}r = Id$$

$$T_{e1}r = Fr - Id$$

$$T_{e1} = \frac{Fr - Id}{r}$$



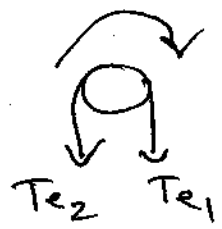
$m_p \neq 0$  pulley has mass  $m_p$ .  
 $I \neq 0$ .



$$T_{e2} > M_2g$$

$$T_{e2} - M_2g = M_2a$$

Force equation

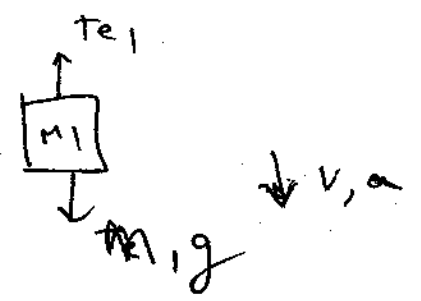


$$T_{e2} \neq T_{e1}$$

$$T_{e1}r > T_{e2}r$$

$$T_{e1}r - T_{e2}r = Id$$

Torque equation



$$M_1g > T_{e1}$$

$$M_1g - T_{e1} = M_1a$$

Force equation

$$T_2 = m_2 a + m_2 g$$

$$T_1 = m_1 g - m_1 a$$

plug into the Torque Equation.

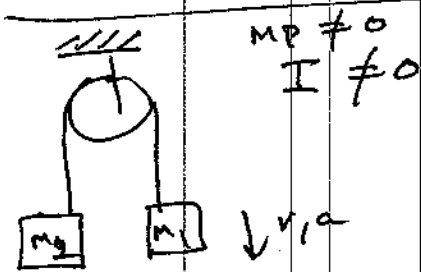
$$r [m_1 g - m_1 a] - r (m_2 a + m_2 g) = I \alpha$$

$$m_1 g - m_2 g = \frac{I \alpha}{r} + m_1 a + m_2 a$$

$$(m_1 - m_2) g = \left( \frac{I}{r^2} + m_1 + m_2 \right) a$$

$$r = \frac{a}{\alpha}$$

## Energy Equivalent Approach



$$m_1 g > m_2 g$$

$$m_1 g - m_2 g = M_e a$$

↑  
Mass equivalent  
 $M_e$

Balance the ~~energy~~ <sup>KE</sup> of the system.

$$\frac{1}{2} M_e v^2 = \frac{1}{2} m_2 v^2 + \frac{1}{2} m_1 v^2 + \frac{1}{2} I \omega^2 \quad (v = r\omega)$$

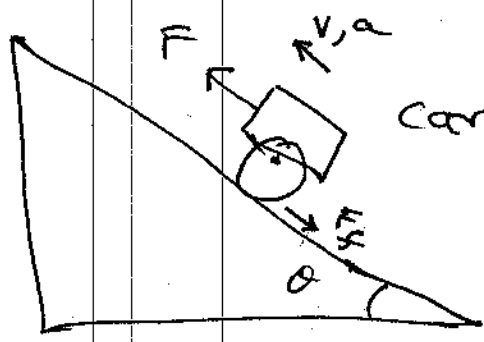
KE of the whole system

$$= \frac{1}{2} m_2 v^2 + \frac{1}{2} m_1 v^2 + \frac{1}{2} \frac{I}{r^2} v^2$$

$$\frac{1}{2} M_e v^2 = \frac{1}{2} \left[ m_2 + m_1 + \frac{I}{r^2} \right] v^2$$

$$m_1 g - m_2 g = \left( m_1 + m_2 + \frac{I}{r^2} \right) a \quad M_e$$





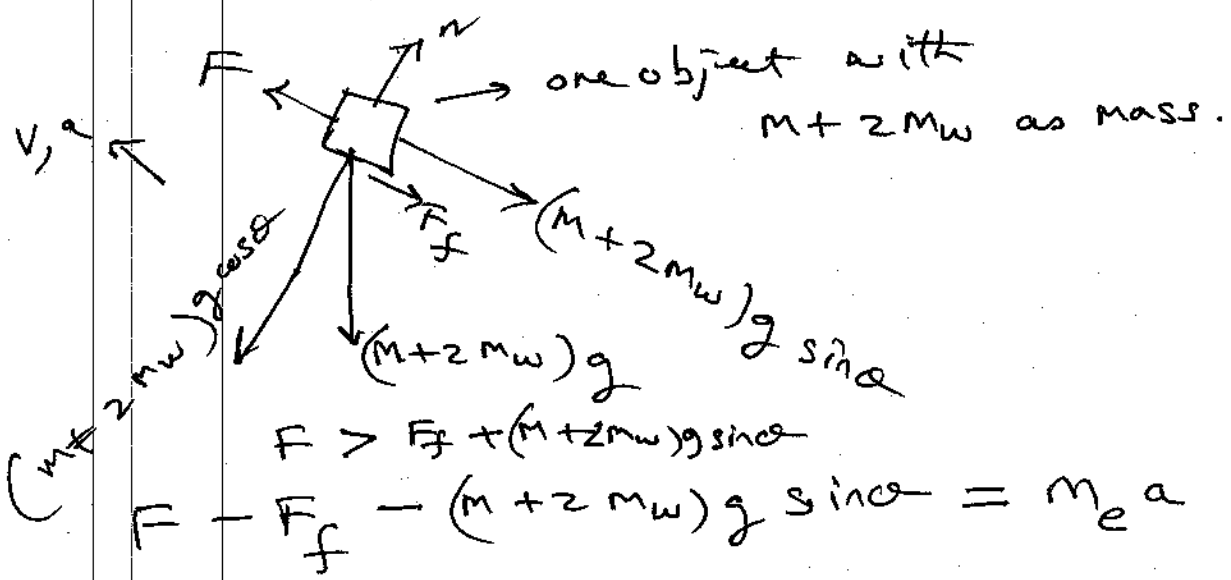
cart with 2 wheels.

mass of cart = m  
mass of wheel = m<sub>w</sub>

Find an equation for acceleration a.

Use energy approach (much easier)

Treat the wheels + the cart as one object.



KE of whole system

$$\frac{1}{2} m_e v^2 = \frac{1}{2} m v^2 + 2 \times \frac{1}{2} m_w v^2 + 2 \times \frac{1}{2} I \omega^2$$

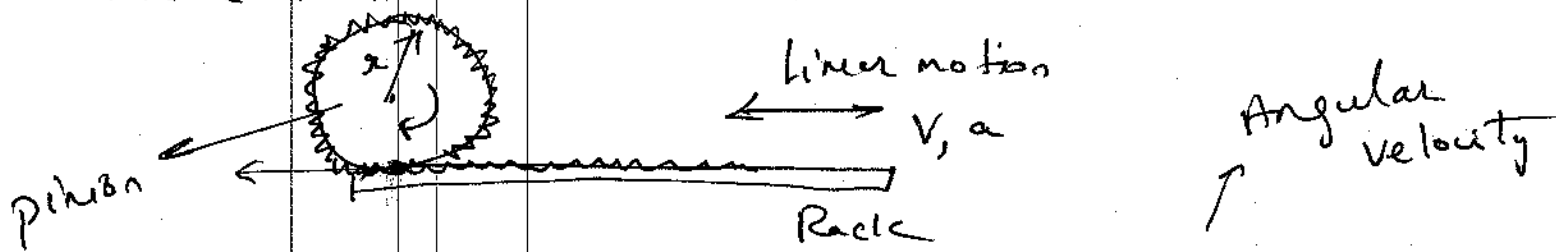
Cart
wheel linear
wheel rotation

$$\frac{1}{2} m_e v^2 = \frac{1}{2} \left[ m + 2m_w + \frac{2I}{r^2} \right] v^2$$

$\underbrace{\hspace{15em}}_{m_e}$

$$F - F_f - (M + 2m_w)g \sin \theta = \left( m + 2m_w + \frac{2I}{r^2} \right) a$$

# Rack & Pinion



Pinion is fixed in position.  $\omega$ ,  $m_p$ ,  $r$

Rack

$M_r$ ,  $v$ ,  $a$

↑ mass of rack  
↑ velocity of rack

↓ mass of pinion

A Torque  $T$  is applied to the pinion.  
Find the equation for acceleration  $a$  of the Rack.

$$T = I_e \alpha$$

With the rack meshed with the pinion.

$$\text{without the rack } T = I \alpha$$

$$\underline{\underline{v = r\omega}}$$

$$\underline{\underline{a = r\alpha}}$$

Rotational  
KE of the whole system

$$\frac{1}{2} I_e \omega^2 = \frac{1}{2} I \omega^2 + \frac{1}{2} M_r v^2$$

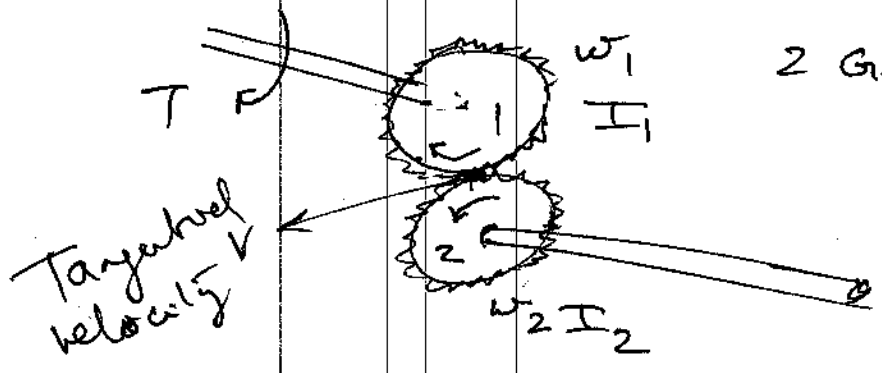
$$\frac{1}{2} I_e \omega^2 = \frac{1}{2} I \omega^2 + \frac{1}{2} M_r r^2 \omega^2$$

$$= \frac{1}{2} \left[ I + M_r r^2 \right] \omega^2$$

$$T = \left[ I + M_r r^2 \right] \alpha$$

$$T = \left[ I + M_r r^2 \right] \frac{a}{r}$$

$I_e$



2 Gears & they are meshed.

T is Torque supplied to Gear 1

Gear ratio  $N = \frac{\omega_1}{\omega_2} = \frac{\text{\# of Teeth on gear 2}}{\text{\# of Teeth on gear 1}}$

$T = I_1 \alpha_1$  if not meshed

larger  $\frac{5}{6} = \frac{20}{40}$  → # of Teeth on smaller gear  
 ↑  
 smaller on larger gear

$T = I_e \alpha_1$  if meshed.

KE ⇒  $\frac{1}{2} I_e \omega_1^2 = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2$

$= \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \frac{\omega_1^2 r_1^2}{r_2^2}$

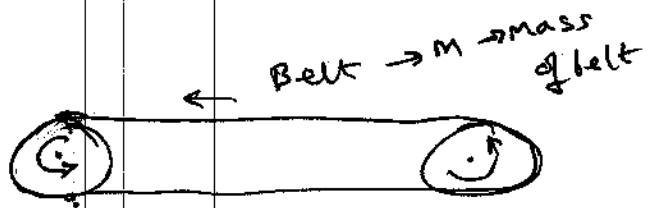
$= \frac{1}{2} \left[ I_1 + \frac{I_2 r_1^2}{r_2^2} \right] \omega_1^2$

$I_e$

$V = r_1 \omega_1$   
 $V = r_2 \omega_2$   
 $\omega_1 r_1 = \omega_2 r_2$   
 $\omega_2 = \frac{\omega_1 r_1}{r_2}$

(OR)  $= \frac{1}{2} \left[ I_1 + \frac{I_2}{N^2} \right] \omega_1^2$

$\frac{1}{N^2} = \frac{r_1^2}{r_2^2}$   
 $N = \frac{r_2}{r_1} = \frac{\omega_1}{\omega_2}$



2 pulleys & a belt

pulley position is fixed.

$\omega_1$   
 $r_1$   
 $T \rightarrow$  Torque is applied.

$\omega_2$   
 $r_2$

$$T = I_e \alpha_1$$

KE

$$\frac{1}{2} I_e \omega_1^2 = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} M v^2$$

$$= \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \frac{r_1^2 \omega_1^2}{r_2^2} + \frac{1}{2} m r_1^2 \omega_1^2$$

$$= \frac{1}{2} \left[ I_1 + \frac{I_2 r_1^2}{r_2^2} + m r_1^2 \right] \omega_1^2$$

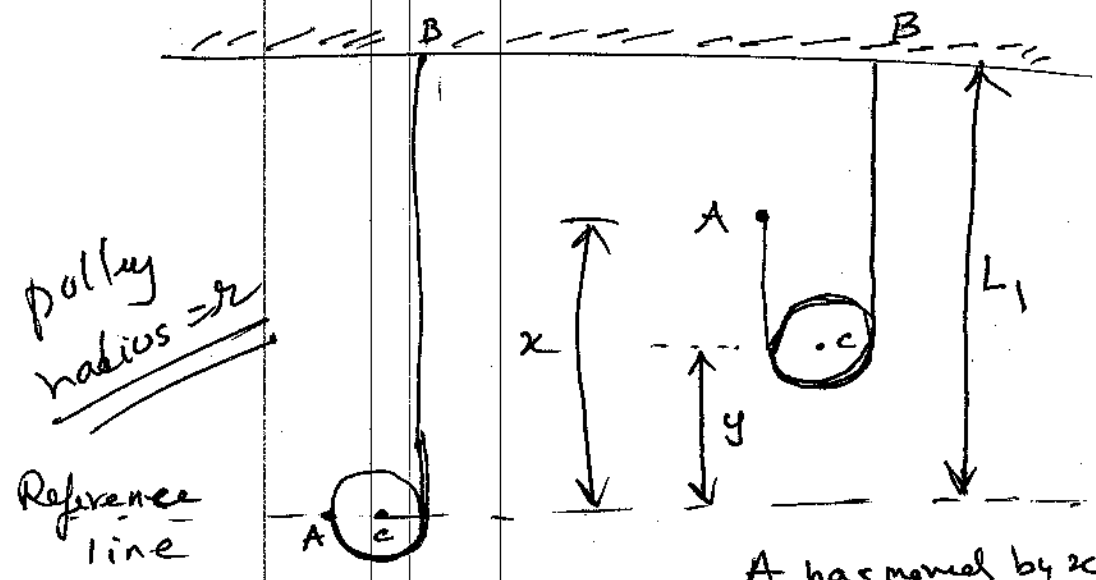
$I_e$

$$v = r_1 \omega_1$$

$$v = r_2 \omega_2$$

$$r_1 \omega_1 = r_2 \omega_2$$

$$\omega_2 = \frac{r_1 \omega_1}{r_2}$$



pulley radius = r

Reference line

A-B is a string.  
↔ pulley.

pulley is NOT FIXED

String wraps around half the perimeter of the pulley

A has moved by x  
pulley C has moved by y

L = length of string  
A to B is L

What is the relation between x & y?

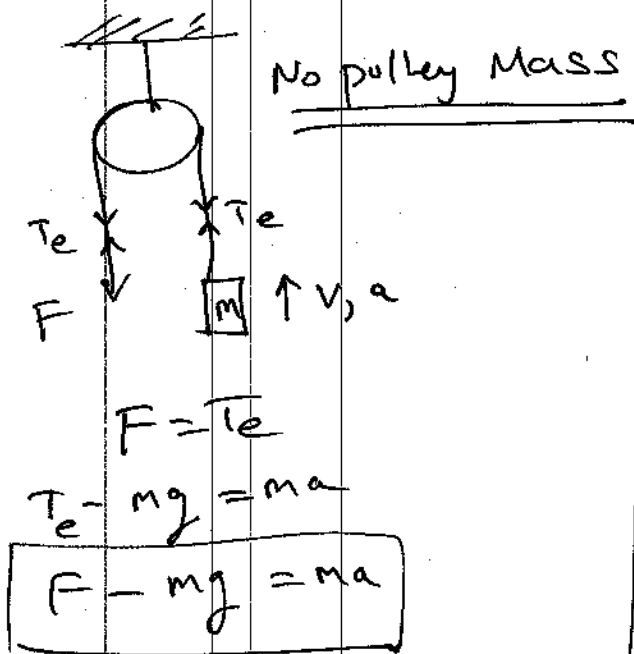
length of string  $L = L_1 + \pi r$

length of string  $L = (x - y) + \pi r + (L_1 - y)$

~~$L + \pi r = (x - y) + \pi r + (L_1 - y)$~~

$x = 2y$

$y = x/2$



Example.

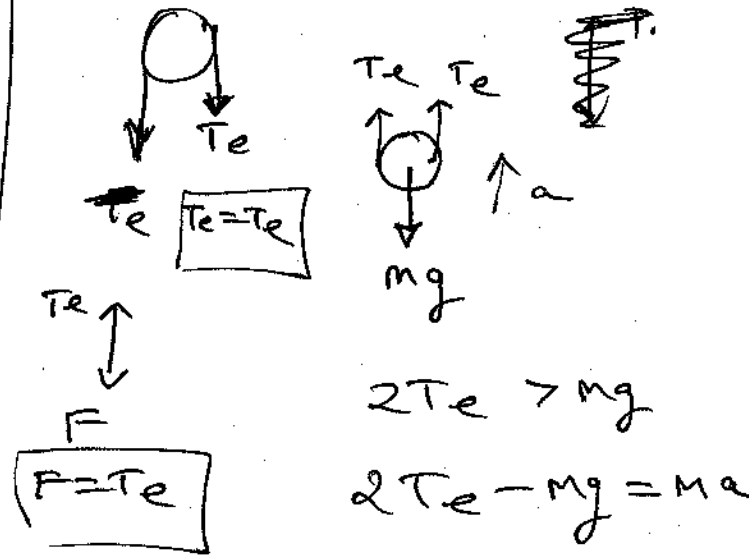
$$m = 1 \text{ kg}$$

$$g = 9.8 \text{ m/s}^2 \approx 10 \text{ m/s}^2$$

$$F = 20 \text{ N}$$

$$20 - (1 \times 10) = 1 \times a$$

$$a = 10 \text{ m/s}^2$$



$$F = \frac{ma + mg}{2}$$

$$F - \frac{mg}{2} = \frac{ma}{2}$$

$$a = 10 \text{ m/s}^2$$

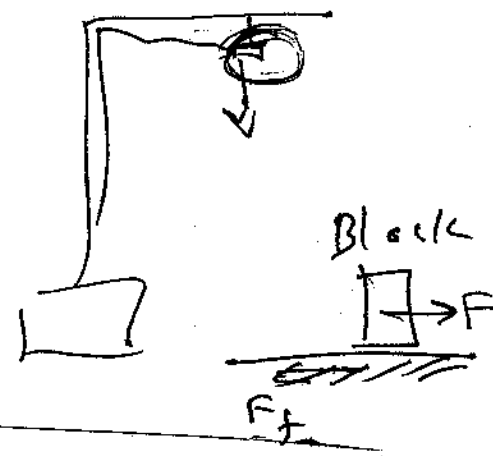
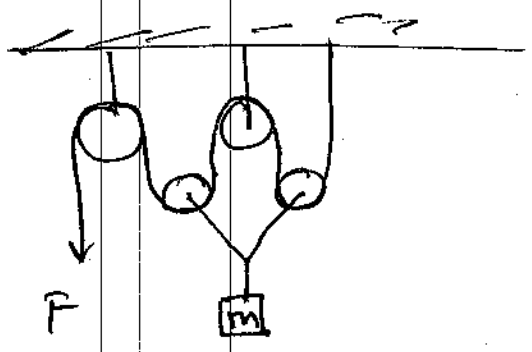
$$m = 1 \text{ kg}$$

$$g = 10 \text{ m/s}^2$$

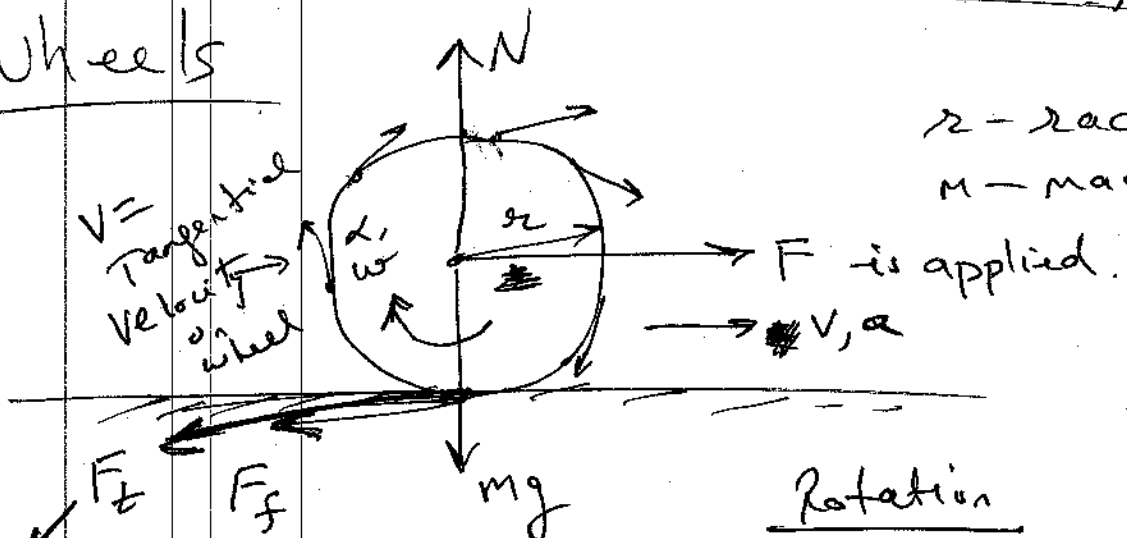
$$F = \frac{1 \times 10}{2} = \frac{1 \times 10}{2}$$

$$F = 5 + 5 = 10 \text{ N}$$

To reduce ~~to~~ to  $1/4$  ~~of~~  $F$ .



Wheels



$r$  - radius  
 $m$  - mass of wheel

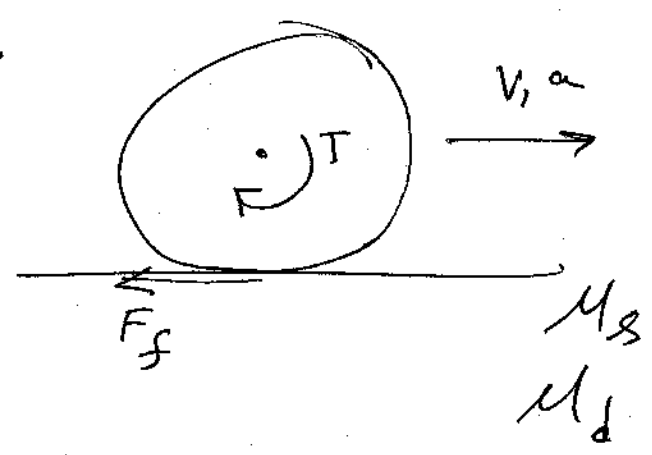
Rotation

$F_f$  dominates over  $F_t$   
 $v = r\omega$

Torque  $T = F_f r$

Tangential Force  
 (exists only for wheels)

only one of them will dominate



$F_f = \mu_s N \rightarrow$  static  
 $F_f = \mu_d N \rightarrow$  dynamic  
 $\mu_d \ll \mu_s$

Rotating:

$v = r\omega$

$F_f > F_t$

← dominates

$F - F_f = ma$

$N = mg$

$F_f = \mu_s N = \mu_s mg$

↑ coefficient of static friction

Sliding

$\omega = 0$

$v \neq 0$

(once) wheels are locked

$F_f = F_t$

Slipping (Some rotation & sliding)

$\omega \neq 0$

$v \neq 0$

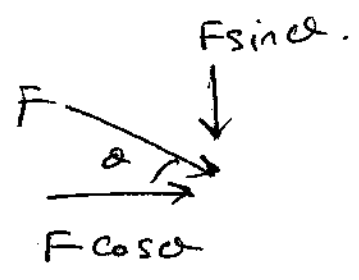
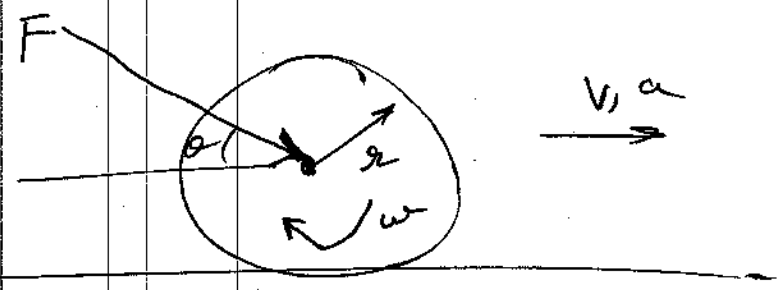
but  $v \neq r\omega$ .

$F_f < F_t$

↑ dominates

$T = \text{Torque} = I\alpha$

$T = F_f r$



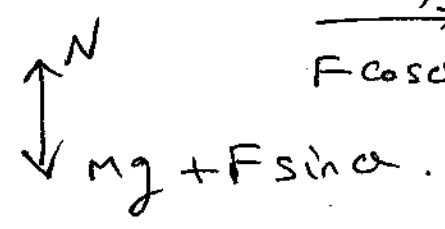
$F \cos \alpha > F_f$

$F \cos \alpha - F_f = ma$

$F \cos \alpha - \mu_s N = ma$

$F \cos \alpha - \mu_s (mg + F \sin \alpha) = ma$

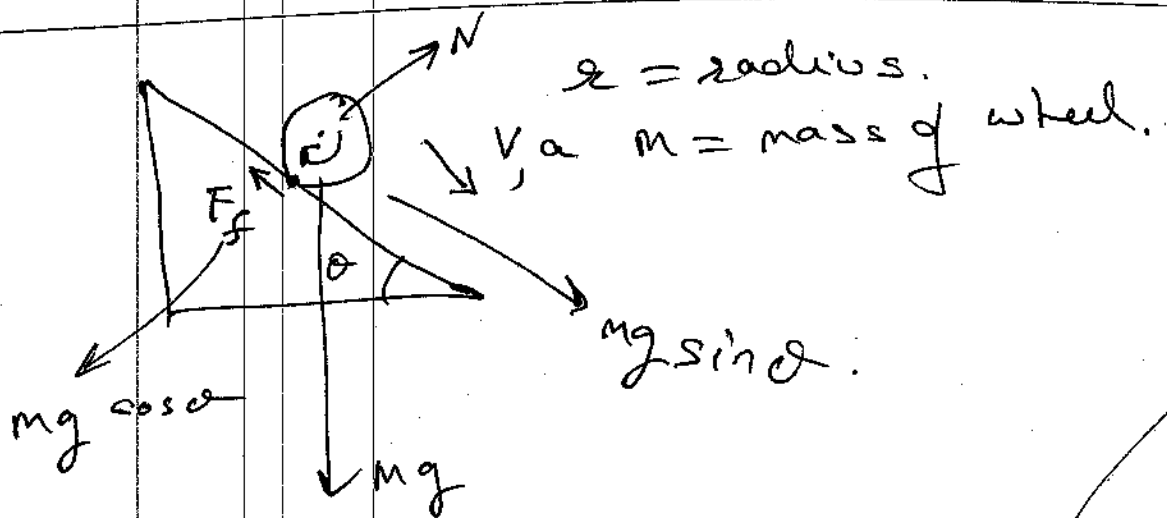
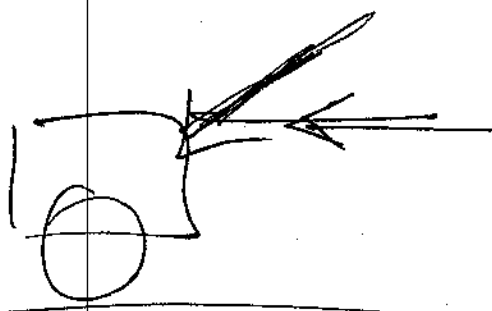
Torque =  $I\alpha = F_f r$



$N = mg + F \sin \alpha$

$F_f = \mu_s N$





$r = \text{radius.}$

$m = \text{mass of wheel.}$

$$mg \sin \theta - F_f = ma$$

$$N = mg \cos \theta$$

$$F_f = \mu_s N$$

$$T = I \alpha = F_f r$$

$$T = I \frac{a}{r} = F_f r$$

$$\frac{1}{2} m r^2 \frac{a}{r} = F_f r$$

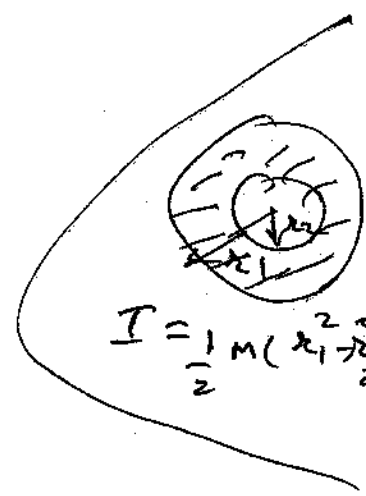
$$F_f = \frac{1}{2} ma$$

$$mg \sin \theta - \frac{1}{2} ma = ma$$

$$mg \sin \theta = \frac{3}{2} ma$$

$$g \sin \theta = \frac{3}{2} a$$

$$a = \frac{2}{3} g \sin \theta$$



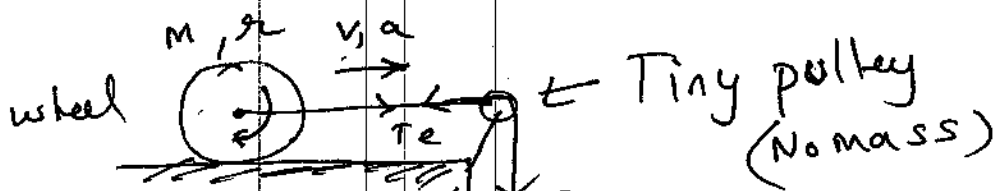
$$I = \frac{1}{2} m r^2$$

$$v = r \omega$$
$$a = r \alpha$$

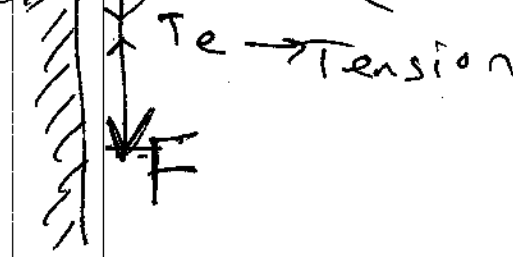
$$I = \frac{1}{2} m r^2$$

For a solid wheel

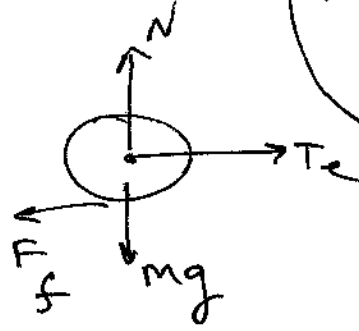
Fact



$F = T_e$



$T_e = T_e$



Rotation  
( $F_f > F_t$ )

~~$T_e$~~  -  $F_f = ma$

$N = mg$

$F_f = \mu_s N$   
 $= \mu_s mg$

$F - F_f = ma$

Torque =  $I\alpha$

$F - \frac{Ia}{r^2} = ma$

$I\alpha = F_f r$

$F_f = \frac{I\alpha}{r}$

$F = \left(m + \frac{I}{r^2}\right)a$

$\alpha = a/r$

$F_f = \frac{Ia}{r^2}$

Energy method

$F = ma$  (In the vertical direction)

↑ mass equivalent

$v = r\omega$

$\omega = \frac{v}{r}$

$\frac{1}{2} m v^2 = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} m v^2 + \frac{1}{2} \frac{I v^2}{r^2}$   
 $= \frac{1}{2} \left[ m + \frac{I}{r^2} \right] v^2$

$$m_e = \frac{m + I}{r^2}$$

$$F = m_e a \quad F = \left[ \frac{m + I}{r^2} \right] a$$