Chapter 5 Block Diagrams

Diagram representation of an ODE.

$$\boxed{T.F. = \frac{X(s)}{F(s)}}$$

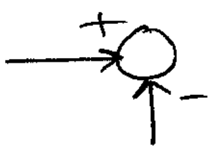
$$F(s) \longrightarrow \boxed{TF} \longrightarrow X(s)$$

input — system state — in the s-domain

Symbols

$\longrightarrow$ arrow

$\square$ → block



summer circle

→ Take-off point

$$F(s) \longrightarrow \boxed{K} \longrightarrow X(s)$$

K is a constant. = T.F.

$$F(s)\,K = X(s)$$

$$\boxed{K = \frac{X(s)}{F(s)}}$$

$\boxed{K}$ → multiplier block

Integrator block $\boxed{1/s}$

$$F(s) \longrightarrow \boxed{1/s} \longrightarrow X(s)$$

$$T \cdot F = 1/s = \frac{X(s)}{F(s)}$$

$$F(s)\frac{1}{s} = X(s)$$

$$F(s) = s\,X(s) \qquad x(0)=0$$

$$f(t) = \dot{x}$$

$$f(t) = \frac{dx}{dt}$$

$$x(t) = \int f(t)\,d(t)$$

$$F(s) \longrightarrow \boxed{K_1} \xrightarrow{B(s)} \boxed{K_2} \longrightarrow \cancel{X}(s) \rightarrow \text{system state}$$

$F(s)$ input

$B(s)$ ↑ intermediate state.

K_1 ~~&~~ K_2 are constants.

Find $\frac{X(s)}{F(s)}$

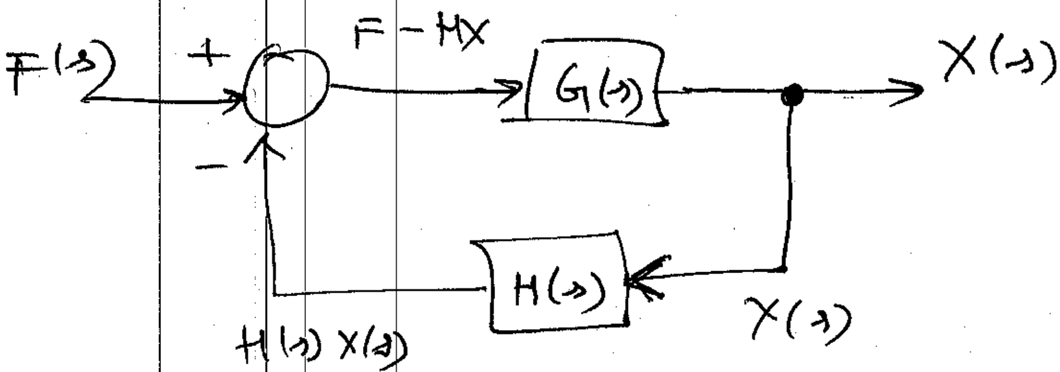
$$F(s)\,K_1 = B(s)$$

$$B(s)\,K_2 = X(s)$$

$$F(s)\,K_1 = \frac{X(s)}{K_2}$$

$$\boxed{\frac{X(s)}{F(s)} = K_1K_2}$$

$$F(s) \longrightarrow \boxed{K_1K_2} \longrightarrow X(s)$$



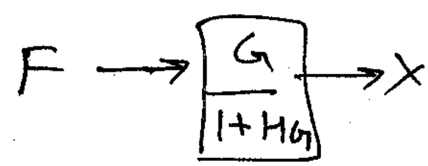
Find $\frac{X}{F}$

$$(F - HX)(G) = X$$

$$FG - HGX = X$$

$$FG = (HG + 1)X$$

$$\boxed{\frac{X}{F} = \frac{G}{1 + HG}}$$



Draw the block diagram

$$\dot{x} + 7x = f(t)$$

By default $x(0) = 0$

Take Laplace Transform on both sides

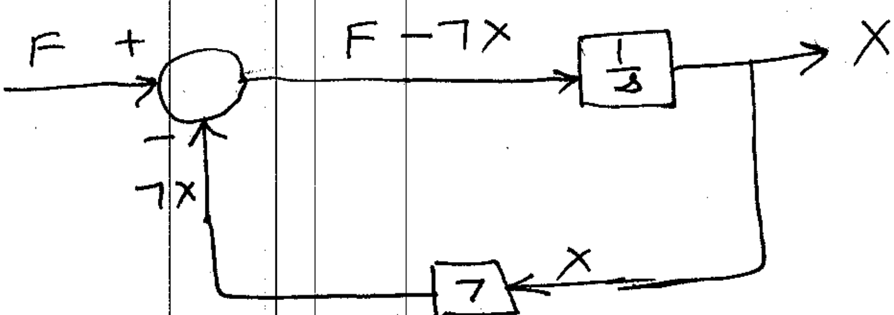
$$s X(s) - \cancel{x(0)}^{\,0} + 7 X(s) = F(s)$$

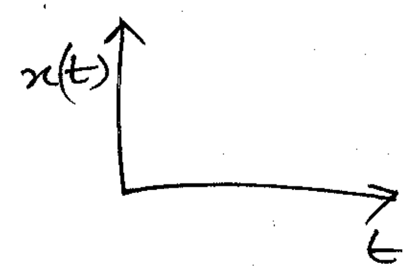
$$s X(s) + 7 X(s) = F(s)$$

Keep the highest power of s to the left and all other terms to the right.

$$s X(s) = F(s) - 7 X(s)$$

$$X(s) = \frac{1}{s} \left[F(s) - 7 X(s) \right]$$





Draw the block Diagram

$$\ddot{x} + 7\dot{x} + 10x = f(t) \qquad \begin{aligned} x(0) &= 0 \\ \dot{x}(0) &= 0 \end{aligned}$$

Take Laplace on both sides.

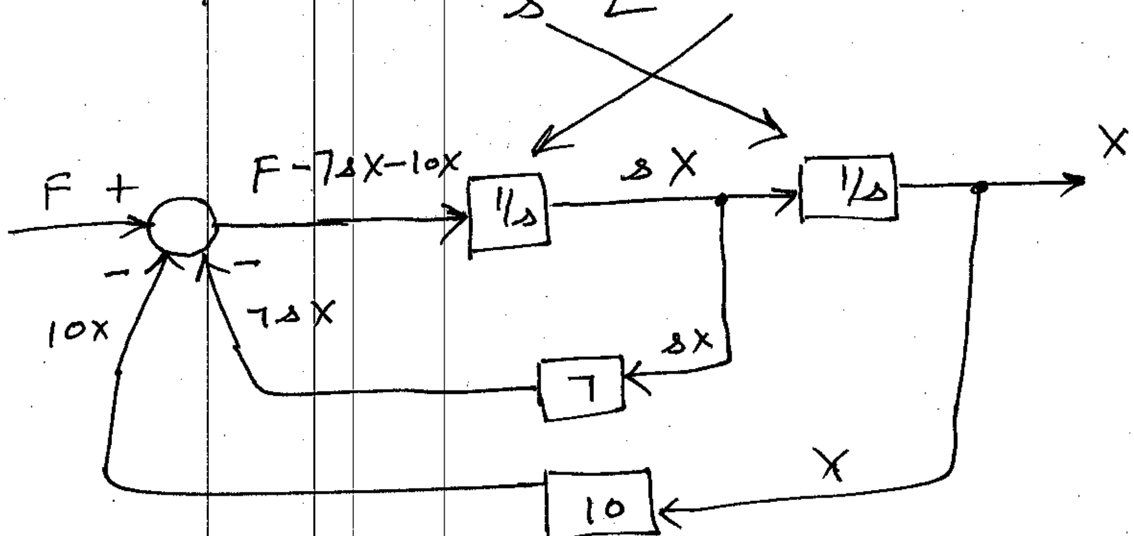
$$s^2X(s) - s\cancel{x(0)}^{\,0} - \cancel{\dot{x}(0)}^{\,0} + 7\left[sX(s) - \cancel{x(0)}^{\,0}\right]$$

$$+ 10X(s) = F(s)$$

$$s^2X(s) + 7sX(s) + 10X(s) = F(s)$$

$$s^2X(s) = F(s) - 7sX(s) - 10X(s)$$

$$X(s) = \frac{1}{s}\left[\frac{1}{s}\left(F - 7sX - 10X\right)\right]$$



Draw the B.D.

By default

$x(0) = 0$

$w(0) = 0$

$$\dot{x} = -3w + f(t)$$
$$\dot{w} = -5w + 4x + g(t)$$

Coupled equations: Two state variables x and w

Two inputs $f(t)$ & $g(t)$

The final state is w

x is an intermediate state.

The Total power is $2 \rightarrow$ sum of the power of each ODE.

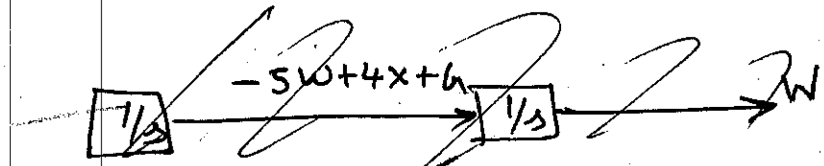
Therefore there are two integrator blocks.

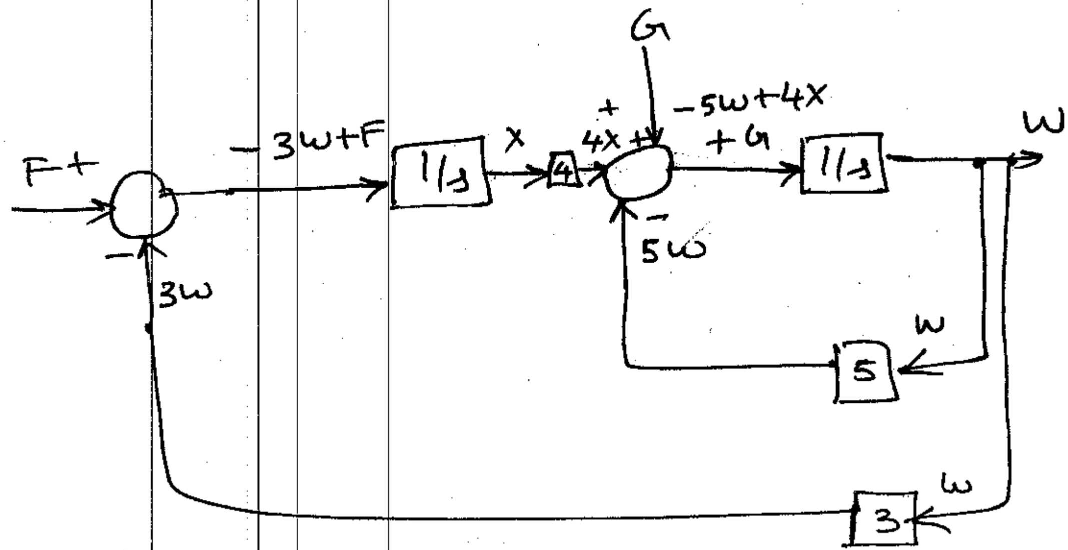
$$sX(s) = -3W(s) + F(s)$$
$$sW(s) = -5W(s) + 4X(s) + G(s)$$

$$X(s) = \frac{1}{s}(-3W + F)$$

$$W = \frac{1}{s}(-5W + 4X + G)$$

a





$\dot{x} + 10x = f(t) + 2g(t)$

Draw the B.D. $x(0) = 0$.

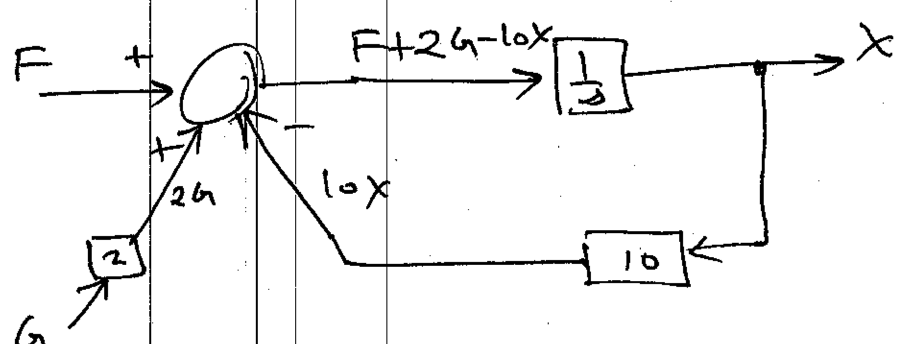
2 inputs.

1 state variable x

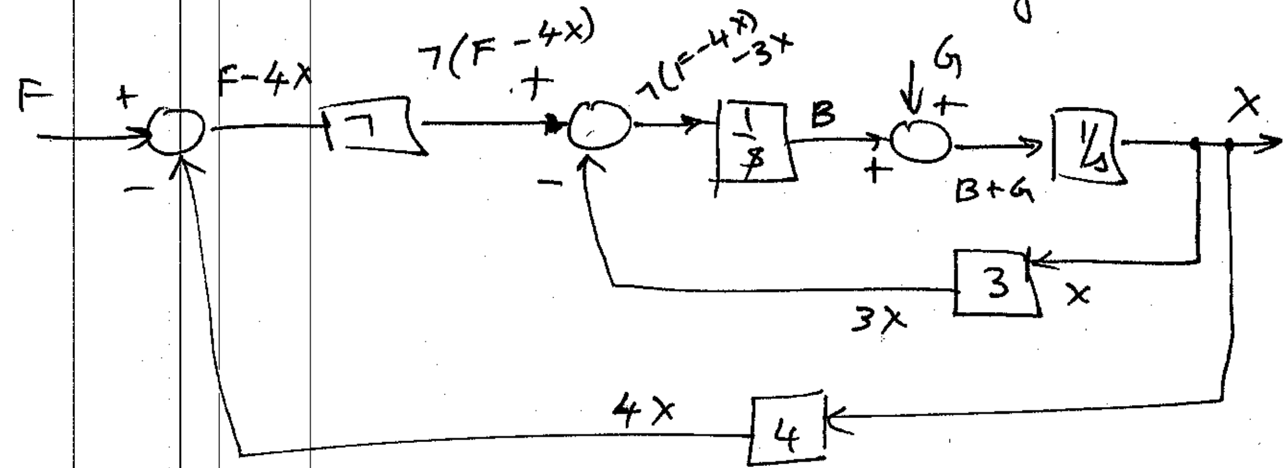
$sX(s) + 10X(s) = F(s) + 2G(s)$

$sX = F + 2G - 10X$

$X = \frac{1}{s}(F + 2G - 10X)$



Finding the model for a given Block Diagram.



F & G are inputs

X → system state variable

$$\left[7(F-4X)-3X\right]\frac{1}{s} = B$$

$$\left[B+G\right]\frac{1}{s} = X$$

$$\left[\left[7(F-4X)-3X\right]\frac{1}{s} + G\right]\frac{1}{s} = X$$

$$\left[\left[7F-28X-3X\right]\frac{1}{s} + G\right]\frac{1}{s} = X$$

$$\left\{7F-31X\right\}\frac{1}{s} + G = sX$$

$$\left[7F-31X\right]\frac{1}{s} = sX-G$$

$$7F-31X = s^2X - Gs$$

$$s^2X+31X = 7F+sG$$

By default $x(0) = 0 \quad \dot{x}(0) = 0$

Taking L^{-1} $\quad \ddot{x} + 31\,x(t) = 7 f(t) + \dot{g}$

Converting higher order ODE into **coupled** **first** order ODE.

$$5\ddot{z} + 7\dot{z} + 4z = f(t)$$

Convert into first order ODEs.

Highest order is 2 $\rightarrow$ we expect two first order ODES $\rightarrow$ with two new state variables.

Let x_1 and x_2 be the two new state variables

Set $x_1 = \dot{z} = \dot{x}_2$ $\quad \boxed{\dot{x}_2 = x_1} \rightarrow \boxed{\dot{x}_2 - x_1 = 0}$

$x_2 = z \Rightarrow \dot{x}_2 = \dot{z}$

$\dot{x}_1 = \ddot{z}$

$$5\dot{x}_1 + 7x_1 + 4x_2 = f(t)$$

Matrix Representation

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 7 & 4 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} f(t)$$

$[2 \times 2]\ [2 \times 1] \qquad [2 \times 1]$

$[1 \times$

$$5\ddot{x}_1 + 12\dot{x}_1 + 5x_1 - 8\dot{x}_2 - 4x_2 = 0$$

$$3\ddot{x}_2 + 8\dot{x}_2 + 4x_2 - 8\dot{x}_1 - 4x_1 = f(t)$$

Find 1st order ODEs.

$$z_1 = \dot{x}_1 = \dot{z}_2 \qquad z_3 = \dot{x}_2 = \dot{z}_4$$

4 new state variables.

$$z_2 = x_1 \qquad z_4 = x_2$$

$$\dot{z}_2 = z_1 \qquad \boxed{\dot{z}_2 - z_1 = 0} \qquad \boxed{\dot{z}_4 - z_3 = 0}$$

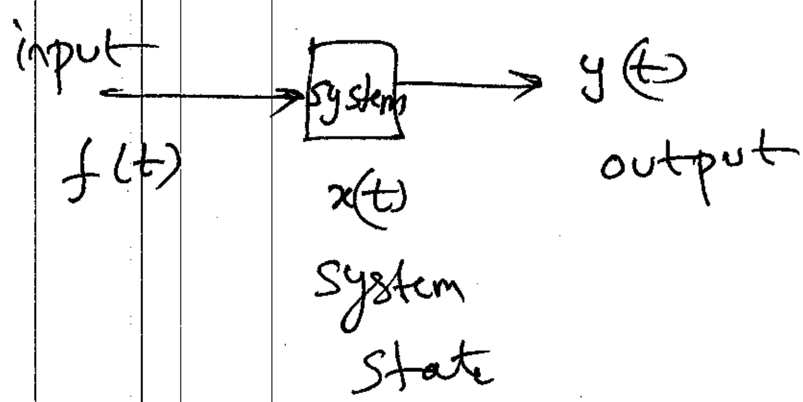
$$\dot{z}_1 = \ddot{x}_1 \qquad \dot{z}_3 = \ddot{x}_2$$

$$5\dot{z}_1 + 12z_1 + 5z_2 - 8z_3 - 4z_4 = 0$$

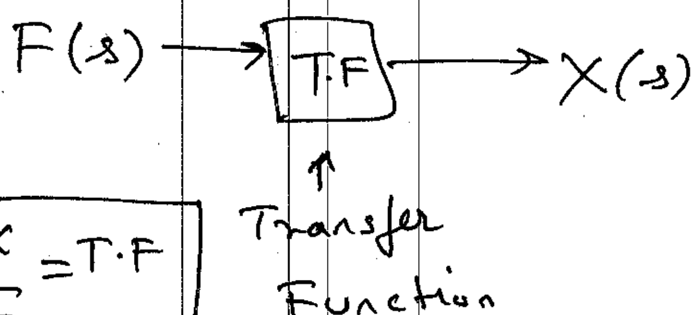
$$3\dot{z}_3 + 8z_3 + 4z_4 - 8z_1 - 4z_2 = f(t)$$

Matrix format

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} + \begin{bmatrix} 12 & 5 & -8 & -4 \\ -1 & 0 & 0 & 0 \\ -8 & -4 & 8 & 4 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} f(t)$$



In the Time Domain



$$\boxed{\frac{X}{F} = T \cdot F}$$

State – Variable Models

s-v models

ODE in standard form.

$$\boxed{\begin{aligned} \dot{x} &= A\, x(t) + B\, f(t) \\ y(t) &= C\, x(t) + D\, f(t) \end{aligned}}$$

state equation

output equation

$$\frac{\text{output}}{\text{input}} = \frac{Y(s)}{F(s)} = \frac{5s+3}{s+2}$$

Express in s-v form.

<u>Two methods</u>.

<u>Method 1</u>. $\dfrac{Y(s)}{F(s)} = \dfrac{5s+3}{s+2}$

highest power of s in the denominator should be made $= 1$

$$\frac{Y}{F} = \frac{\frac{5s}{s} + \frac{3}{s}}{\frac{s}{s} + \frac{2}{s}}$$

$$\frac{Y}{F} = \frac{5 + \frac{3}{s}}{1 + \frac{2}{s}}$$

$$Y\left[1 + \frac{2}{s}\right] = F\left[5 + \frac{3}{s}\right]$$

$$Y + \frac{2Y}{s} = 5F + \frac{3F}{s}$$

$$Y = 5F + \frac{3F}{s} - \frac{2Y}{s}$$

$$Y = 5F + \frac{1}{s}\underbrace{\left[3F - 2Y\right]}_{X}$$

$X \longrightarrow$ Let this term equal the state Variable

Let $X = \frac{1}{s}\left[3F - 2Y\right]$

$sX = 3F - 2Y$

L^{-1} $\boxed{\dot{x} = 3f(t) - 2y(t)}$

$Y = 5F + X$

L^{-1} $\boxed{y(t) = x(t) + 5f(t)}$

$\dot{x} = 3f(t) - 2\left[x(t) + 5f(t)\right]$

$\dot{x} = -7f(t) - 2x(t)$

$\boxed{\dot{x} = -2x(t) - 7f(t)}$

$A = -2 \quad B = -7$

$c = 1 \quad D = 5$

output equation

state equation

S-V FORM

$$\frac{Y}{F} = \frac{5s+3}{s+2}$$

Express in S-V form.

Method 2

$$Y = (5s+3)\underbrace{\left[\frac{F}{s+2}\right]}_{X}$$

combine the input & the denominator as one term

Let $X = \frac{F}{s+2}$

$$sX + 2X = F$$

$$sX = -2X + F$$

$$\mathcal{L}^{-1} \quad \boxed{\dot{x} = -2x(t) + f(t)}$$

$$Y = (5s+3)X$$

$$Y = 5sX + 3X$$

$$\mathcal{L}^{-1} \quad y(t) = 5\dot{x} + 3x(t)$$

$$= 5\left(-2x(t) + f(t)\right) + 3x(t)$$

$$y(t) = -10x(t) + 3x(t) + 5f(t)$$

$A = -2$, $B = 1$, $C = -7$, $D = 5$

$$\boxed{y(t) = -7x(t) + 5f(t)}$$

Method 1 is the dual of Method 2 or vice versa

$$\frac{\text{output}}{\text{input}} = \frac{Y}{U} = \frac{4s+7}{5s^2+4s+7}$$

Express in s-v form.

Method 1

Divide by $5s^2$

$$\frac{Y}{U} = \frac{\frac{4s}{5s^2} + \frac{7}{5s^2}}{1 + \frac{4s}{5s^2} + \frac{7}{5s^2}} = \frac{\frac{4}{5s} + \frac{7}{5s^2}}{1 + \frac{4}{5s} + \frac{7}{5s^2}}$$

$$Y\left[1 + \frac{4}{5s} + \frac{7}{5s^2}\right] = U\left[\frac{4}{5s} + \frac{7}{5s^2}\right]$$

$$Y = \frac{4U}{5s} + \frac{7U}{5s^2} - \frac{4Y}{5s} - \frac{7Y}{5s^2}$$

$$Y = \underbrace{\frac{1}{s}\left[\left(\frac{4U}{5} - \frac{4Y}{5}\right) + \underbrace{\frac{1}{s}\left(\frac{7U}{5} - \frac{7Y}{5}\right)}_{x_1}\right]}_{x_2}$$

Let $x_1 = \frac{1}{s}\left[\frac{7U}{5} - \frac{7Y}{5}\right]$

$$s x_1 = \frac{7U}{5} - \frac{7Y}{5} \quad \xrightarrow{\mathcal{L}^{-1}} \quad \dot{x}_1 = \frac{7}{5}u(t) - \frac{7}{5}y(t)$$

$$\dot{x}_1 = \frac{7}{5}u(t) - \frac{7}{5}x_2(t)$$

$$Y = X_2$$

$$L^{-1} \Rightarrow \boxed{y(t) = x_2(t)}$$

$$X_2 = \frac{1}{s}\left[\frac{4U}{5} - \frac{4Y}{5} + X_1\right]$$

$$sX_2 = \frac{4U}{5} - \frac{4Y}{5} + X_1$$

$$L^{-1} \Rightarrow \dot{x}_2 = \frac{4}{5}u(t) - \frac{4}{5}y(t) + x_1(t)$$

$$\dot{x}_2 = \frac{4}{5}u(t) - \frac{4}{5}x_2(t) + x_1(t)$$

Rewriting

$$\dot{x}_1 = -\frac{7}{5}x_2(t) + \frac{7}{5}u(t)$$

$$\dot{x}_2 = +x_1(t) - \frac{4}{5}x_2(t) + \frac{4}{5}u(t)$$

$$y = x_2(t)$$

S-V
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$A = \begin{bmatrix} 0 & -\frac{7}{5} \\ 1 & -\frac{4}{5} \end{bmatrix} \quad B = \begin{bmatrix} 7/5 \\ 4/5 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad D = 0$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0\begin{bmatrix} u \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{7}{5} \\ 1 & -4/5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Bu$$

Method 2

$$\frac{Y}{U} = \frac{4s+7}{5s^2+4s+7}$$

$$Y = 4s+7 \underbrace{\left[\frac{U}{5s^2+4s+7}\right]}_{X_1}$$

$$Y = 4sX_1 + 7X_1$$

$$L^{-1} \quad y(t) = 4\dot{x}_1 + 7x_1(t)$$

$$\boxed{y(t) = 4x_2 + 7x_1}$$

$$X_1 = \frac{U}{5s^2+4s+7}$$

$$5X_1s^2 + 4X_1s + 7X_1 = U$$

$$5X_1s^2 = U - 4X_1s - 7X_1$$

$$X_1 s = \frac{1}{5s}\left[U - 4X_1s - 7X_1\right]$$

$$X_1 s = \frac{1}{s}\underbrace{\left[\frac{U}{5} - \frac{4X_1s}{5} - \frac{7X_1}{5}\right]}_{X_2}$$

$$X_1 s = X_2$$

$$L^{-1} \quad \dot{x}_1 = x_2(t)$$

$$X_2 = \frac{1}{s}\left[\frac{U}{5} - \frac{4X_1s}{5} - \frac{7X_1}{5}\right]$$

$$sX_2 = \frac{U}{5} - \frac{4X_1s}{5} - \frac{7X_1}{5}$$

L^{-1} (18)

$$\dot{x}_2 = \frac{1}{5}u(t) - \frac{4}{5}\dot{x}_1 - \frac{7}{5}x_1(t)$$

$$\boxed{\dot{x}_2 = \frac{1}{5}u(t) - \frac{4}{5}x_2(t) - \frac{7}{5}x_1(t)}$$

Rewrite

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{7}{5}x_1 - \frac{4}{5}x_2 + \frac{1}{5}u$$

$$y = 7x_1 + 4x_2$$

S-V
$\dot{x} = Ax + Bu$
$y = Cx + Du$

$$A = \begin{bmatrix} 0 & 1 \\ -7/5 & -\frac{4}{5} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$$

$$C = \begin{bmatrix} 7 & 4 \end{bmatrix} \quad D = 0$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -7/5 & -4/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/5 \end{bmatrix} [u]$$

$$y = \begin{bmatrix} 7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (0)[u]$$