Stability of ODE

Find the characteristic Roots.

1) If ALL Roots are negative → stable

2) If ANY one Root is positive → unstable

3) If ANY one Root is zero → neutrally Stable (oscillates) (periodic functions like sine / cosine)

[IGNORE COMPLEX PART OF THE ROOT]

$\ddot{x} - 25x = 10$

Is this stable?

$\ddot{x} \to s^2$

$\dot{x} \to s$

$x \to 1$

$s^2 - 25 = 0$ Characteristic equation (IGNORE THE RIGHT SIDE → INPUT)

$s^2 = 25$

$s = \pm 5$

one positive Root

→ UNSTABLE

$\ddot{x} + 25x = 10$

$s^2 + 25 = 0$

$s = \sqrt{-25} = 0 \pm 5j$

↑ ↑ ignore the complex part

The real part is zero → neutrally stable (oscillates)

Transfer Function

input $\longrightarrow$ $\boxed{\text{system } x(t)}$ $\longrightarrow$ output

$u(t)$ or $f(t)$ (Right side of an ODE) $\qquad$ $y(t)$ $\qquad$ [Time Domain]

$$U(s) \text{ or } F(s) \longrightarrow \boxed{\text{Transfer Function T.F.}} \longrightarrow X(s)$$

[s - Domain (Abstract)]

$$\boxed{\text{T.F.} = \frac{X(s)}{F(s)}}$$

Definition of a Transfer function.

By default for T.F. $\quad x(0) = 0$, $\quad \dot{x}(0) = 0$

$$F(s) \longrightarrow \boxed{\frac{1}{s+2}} \longrightarrow X(s)$$

$$\dot{x} + 2x = f(t)$$

Find the T.F.

$$s X(s) - \cancel{x(0)}^{\,0} + 2X(s) = F(s)$$

$$s X(s) + 2X(s) = F(s)$$

$$X(s)(s+2) = F(s)$$

$$\text{T.F} = \frac{X(s)}{F(s)} = \frac{1}{(s+2)}$$

Ex

$$5\ddot{x} + 30\dot{x} + 40x(t) = 6f(t) - 20g(t)$$

Find T.F. $\qquad x(0) = 0 \quad \dot{x}(0) = 0$

There are $\underline{\underline{2}}$ T.F's, one for each input

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$$\frac{X(s)}{F(s)} \quad \text{and} \quad \frac{X(s)}{G(s)}$$

$$5(s^2X(s) - \underbrace{sx(0)}_{0} - \underbrace{\dot{x}(0)}_{0}) + 30(sX(s) - \underbrace{x(0)}_{0}) + 40X(s) = 6F(s) - 20G(s)$$

$$5s^2X(s) + 30sX(s) + 40X(s) = 6F(s) - 20G(s)$$

$$(5s^2 + 30s + 40)X(s) = 6F(s) - 20G(s)$$

$$\frac{X(s)}{F(s)} = \frac{6}{5s^2 + 30s + 40}$$

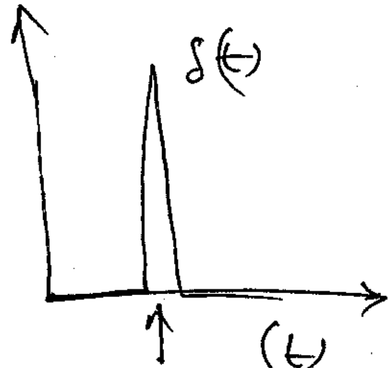
$$F(s) \longrightarrow \boxed{} \longrightarrow X(s)$$

$$\frac{X(s)}{G(s)} = \frac{-20}{5s^2 + 30s + 40}$$

$$G(s) \longrightarrow \boxed{} \longrightarrow X(s)$$

Impulse Function

$\delta(t)$
delta function.



impulse
(lasts for a fraction of a second)

$L\,[\delta(t)] = 1$

$$\boxed{X(s) = \frac{1}{s+5}}$$

$x(0) = 0$

$$\boxed{\text{Find } x(0+) = \lim_{s \to \infty} s X(s)}$$

$$\lim_{s \to \infty} s \left(\frac{1}{s+5} \right) = \lim_{s \to \infty} \frac{\frac{s}{s}}{\frac{s}{s} + \frac{5}{s}} = \frac{1}{\lim_{s \to \infty} 1 + \frac{5}{s}}$$

$$= 1$$

$$\boxed{x(0+) = 1}$$

$$(s+5)\,X(s) = 1$$

$$s\,X(s) + 5\,X(s) = 1$$

Taking L^{-1} $\quad \dot{x}(t) + 5x = \delta(t)$ impulse input

$\ddot{x} = \delta(t) \qquad x(0) = 5$

$\qquad\qquad\qquad \dot{x}(0) = 10$

Find the impact of the impulse by studying $x(0+)$

$\dot{x}(0+)$

$$x(0+) = \lim_{s \to \infty} s X(s)$$

$$\dot{x}(0+) = \lim_{s \to \infty} s \left[s X(s) - x(0) \right]$$

$\ddot{x} = \delta(t)$

Laplace Transform. $s^2 X(s) - s x(0) - \dot{x}(0) = 1$

$$s^2 X(s) - 5s - 10 = 1$$

$$\boxed{X(s) = \frac{5s + 11}{s^2}}$$

$$x(0+) = \lim_{s \to \infty} s X(s) = s\left(\frac{5s + 11}{s^2}\right) = \lim_{s \to \infty} \frac{5s + 11}{s}$$

$$= \lim_{s \to \infty} \frac{\frac{5s}{s} + \frac{11}{s}}{s/s} = \lim_{s \to \infty} = 5$$

$$x(0) = x(0+) = 5$$

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$$\dot{x}(0+) = \lim_{s\to\infty} s\left[s X(s) - x(0) \right)$$

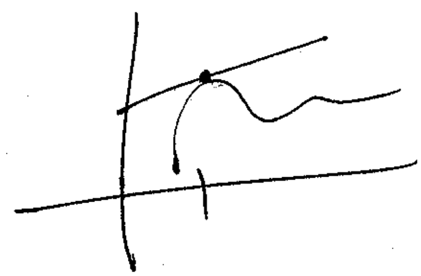
$$= \lim_{s\to\infty} s\left[s\left[\frac{5s+11}{s^2}\right] - 5\right]$$

$$= \lim_{s\to\infty} \cancel{s^2}\left[\frac{5s+11}{\cancel{s^2}}\right] - 5s$$

$$= \lim_{s\to\infty} \left[5s+11-5s\right\} = 11$$

$x(0)=5 \quad | \quad \dot{x}(0) = 10$

$x(0+)=5 \quad | \quad \dot{x}(0+) = 11$



To identify parts of ~~an~~ ODE's solution.

free — depends on initial condition.

forced — depends on input

Steady — stays over time

Transient — vanishes over time

Initial Value Theorem ($\underline{I}$VT)

Given $x(0)$ find $x(0^+)$

IVT does not apply if the (highest) power of s in the numerator & denominator are equal for $X(s)$

$$\boxed{x(0+) = \lim_{s \to \infty} s\,X(s)}$$

$x(0) = 0$

$$X(s) = \frac{7s+2}{s(s+6)} = \frac{7s+2}{s^2+6s}$$

$$x(0+) = \lim_{s \to \infty} s\left(\frac{7s+2}{s^2+6s}\right) = \lim_{s \to \infty} \frac{7s+2}{s+6}$$

$$= \lim_{s \to \infty} \frac{(7s+2)/s}{(s+6)/s} = \lim_{s \to \infty} \frac{7+\frac{2}{s}}{1+\frac{6}{s}} = 7$$

$$x(0+) = 7$$

$$X(s) = \frac{7s+2}{s(s+6)} = \frac{c_1}{s} + \frac{c_2}{s+6}$$

$$c_1 = \lim_{s \to 0} \cancel{s}\left(\frac{7s+2}{\cancel{s}(s+6)}\right) = 2/6 = 1/3$$

$$c_2 = \lim_{s \to -6} \cancel{(s+6)}\left(\frac{7s+2}{s\cancel{(s+6)}}\right) = \frac{-40}{-6} = \frac{20}{3}$$

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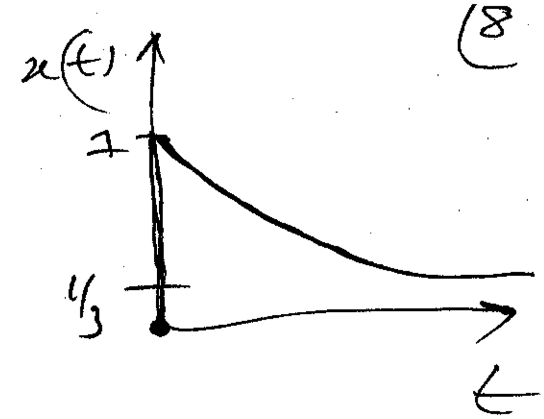
$$\boxed{x(t) = \frac{1}{3} + \frac{20}{3} e^{-6t}}$$



$$X(s) = \frac{s+4}{s+3}$$ Find $x(0+)$

IVT does not apply
because highest power of s
is the same in the
num & den of $X(s)$

$x(0) = 0$

If you apply IVT

$$x(0^+) = \lim_{s \to \infty} s X(s) = \lim_{s \to \infty} \frac{s(s+4)}{s+3} = \lim_{s \to \infty} \frac{s+4}{1 + \frac{3}{s}}$$

(Divide by s)

$$= \infty$$

IVT does not apply (in reality)

$$X(s) = \frac{s+4}{s+3}$$ Take inverse

$$= \frac{s}{s+3} + \frac{4}{s+3} = C_1 + \frac{C_2}{(s+3)}$$

$$s+4 = c_1(s+3) + c_2$$

$s \rightarrow \quad 1 = c_1$

constants $\rightarrow \quad 4 = 3c_1 + c_2$

$$c_2 = 1$$

$$X(s) = 1 + \frac{1}{s+3}$$

$$\boxed{x(t) = \underbrace{\delta(t)}_{\text{impulse}} + e^{-3t}}$$

Final value theorem (FVT)

Given $x(0)$ find $x(\infty)$

$$\boxed{x(\infty) = \lim_{s \to 0} s\,X(s)}$$

FVT does not apply if the real roots of the ~~depo~~ denominator of $sX(s)$ is positive, zero or the function $x(t)$ is periodic.

$$X(s) = \frac{9s+2}{s(s+2)}$$

Roots of $sX(s) = \cancel{s}\left[\frac{9s+2}{\cancel{s}(s+2)}\right]$

$$= \frac{9s+2}{(s+2)}$$

Root here is -2 for $sX(s)$

(FVT applies)

$$x(\infty) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{9s+2}{s+2} = 2/2 = 1$$

$$X(s) = \frac{9s+2}{s^2(s+2)} \qquad sX(s) = \cancel{s}\left(\frac{9s+2}{\cancel{s^2}(s+2)}\right)$$

$$sX(s) = \frac{9s+2}{s(s+2)}$$

Roots $0, -2$

Cannot apply FVT

If you apply

$$x(\infty) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{9s+2}{s(s+2)}$$

$=$ Divide by 0.

Cannot apply FVT

$$X(s) = \frac{5}{s^2+25} \qquad x(\infty) = ?$$

$$s\,X(s) = \frac{5s}{s^2+25}$$

Roots $\Rightarrow s^2 = -25$

$s = 0 \pm 5i$

real root is 0

Cannot apply FVT

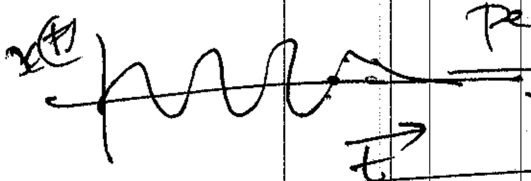
(imaginary is ignored)

$$\lim_{s\to 0} \frac{5s}{s^2+25} = 0 \quad X$$

$$x(t) = L^{-1}\left[\frac{5}{s^2+5^2}\right]$$

$$\boxed{x(t) = \sin 5t}$$

periodic function.



Impulse Function $\delta(t)$
delta function.

$$L(\delta(t)) = 1$$

$x(0) = 0$

$$X(s) = \frac{F(s)}{(s+5)}$$

$$\boxed{\begin{aligned} f(t) &= \delta(t) \\ F(s) &= 1 \end{aligned}}$$

$$(s+5)X(s) = F(s)$$
$$sX(s) + 5X(s) = F(s)$$
$$\dot{x} + 5x = f(t) = \delta(t)$$

$$\boxed{X(s) = \frac{1}{s+5}}$$

$$x(0^+) = \lim_{s\to\infty} s\,X(s) = \frac{s}{\lim_{s\to\infty} s+5} = \lim_{s\to\infty} \frac{s/s}{1+\frac{5}{s}}$$

$$x(0^+) = \lim_{s\to\infty} = 1/1 = 1$$