Q.

$$X(s) = \frac{8s+13}{s^2+4s+53} \qquad \text{Find } x(t)$$

$$= \frac{8s+13}{s^2+4s+4+49}$$

$$= \frac{8s+13}{(s^2+4s+2^2)+7^2} = \frac{8s+13}{(s+2)^2+7^2}$$

$$\frac{8s+13}{(s+2)^2+7^2} = \frac{c_1(s+2)}{(s+2)^2+7^2} + \frac{c_2\,(7)}{(s+2)^2+7^2} \qquad \text{Partial fractions.}$$

$$8s+13 = c_1(s+2) + 7c_2$$

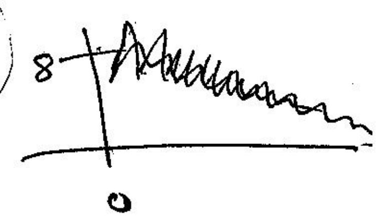
Collect s terms $\quad 8s = c_1 s \quad c_1 = 8$

collect constants $\quad 13 = 2c_1 + 7c_2 \quad c_2 = -3/7$

$$= \frac{8 \times (s+2)}{\left((s+2)^2+7^2\right)} - \frac{3}{7}\left(\frac{7}{(s+2)^2+7^2}\right)$$

Take inverse $\mathcal{L}$

$$x(t) = 8e^{-2t}\cos 7t - \frac{3}{7}e^{-2t}\sin 7t$$



Solving ODE

Solve $\dot{x} + 5x = 3 \quad x(0) = 3$

$$\frac{dx}{dt} + 5\,x(t) = 3 \longrightarrow \text{constant input}$$

Finding $x(t) = ?$

RHS is the input to the ODE.

Solve using Laplace

Take Laplace Transform on both sides.

$$s\,X(s) - x(0) + 5\,X(s) = \frac{3}{s}$$

$$s\,X(s) - 3 + 5\,X(s) = \frac{3}{s}$$

$$X(s)\,(s+5) = \frac{3}{s} + 3$$

$$(s+5)\,X(s) = \frac{3+3s}{s}$$

$$X(s) = \frac{3+3s}{s(s+5)} = \frac{c_1}{s} + \frac{c_2}{s+5}$$

$$c_1 = \lim_{s\to 0} s\,X(s) = \lim_{s\to 0} \cancel{s}\left(\frac{3+3s}{\cancel{s}(s+5)}\right) = 3/5$$

$$c_2 = \lim_{s\to -5} (s+5)\,X(s) = \lim_{s\to -5} \frac{(3+3s)\cancel{(s+5)}}{s\cancel{(s+5)}}$$

$$= \frac{-12}{-5} = 12/5$$

$$X(s) \cancel{x(t)} = \frac{3}{5}\left(\frac{1}{s}\right) + \frac{12}{5}\left(\frac{1}{s+5}\right)$$

Take inverse

$$\boxed{x(t) = \frac{3}{5} + \frac{12}{5} e^{-5t}}$$

$$\dot{x} + 5x = 3 \qquad \underline{x(0) = 3}$$

<u>From Table</u> $\quad \dot{x} + ax = b \qquad x(t) = \frac{b}{a} + ce^{-at}$

$a = 5$
$b = 3$

$$x(t) = \frac{3}{5} + ce^{-5t}$$

$$x(0) = \frac{3}{5} + ce^{0} = 3$$

$$c = 3 - \frac{3}{5} = \frac{12}{5}$$

$$\boxed{x(t) = \frac{3}{5} + \frac{12}{5} e^{-5t}}$$

Solve $\ddot{x} + 7\dot{x} + 10x = 20$ $\qquad x(0) = 5$

$\dot{x}(0) = 3$

$\ddot{x} + a\dot{x} + bx = c$

$a = 7$

$b = 10$

$c = 20$ (input)

Characteristic equation of an ODE.

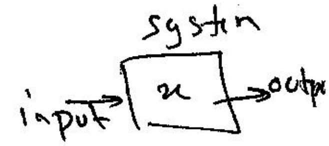
Focus only on Left side of the equation

$\ddot{x} \Rightarrow s^2$

$\dot{x} \Rightarrow s$

$x \rightarrow 1$

$s^2 + 7s + 10 = 0$



Find the roots.

$(s+5)(s+2) = 0$

Roots $-5, -2$ $\quad$ distinct Roots

$(s_1) \quad (s_2)$

Solution from table

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \frac{c}{b}$$

$$x(t) = c_1 e^{-5t} + c_2 e^{-2t} + \frac{20}{10}$$

$$x(0) = 5 = c_1 + c_2 + 2$$

$$c_1 + c_2 = 3$$

$$\dot{x}(t) = -5c_1 e^{-5t} - 2c_2 e^{-2t}$$

$$\dot{x}(0) = 3 = -5c_1 - 2c_2$$

$$c_1 = -3 \qquad c_2 = 6$$

$$x(t) = -3e^{-5t} + 6e^{-2t} + 2$$

Ex

Solve $\ddot{x} + 4\dot{x} + 53x = 15$

$x(0) = 8$ ~~$x(0) = 8$~~

$\dot{x}(0) = -19$

$\ddot{x} + a\dot{x} + bx = c$

$a = 4 \quad b = 53 \quad c = 15$

RHS → constant input (you may use the table).

(2.3.2 page 53) 3rd Ed.

Characteristic equation

$\ddot{x} \to s^2$

$\dot{x} \to s$

$x \to 1$

$s^2 + 4s + 53 = 0$

$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-4 \pm \sqrt{4^2 - (4 \times 1 \times 53)}}{2 \times 1}$$

$$\text{Roots} :- \frac{-4 \pm \sqrt{-196}}{2}$$

$$= -2 \pm 7i$$

Case 4 :- $\sigma \pm i\omega$

$\sigma = -2$

$\omega = 7$

Solution :- from the table

$$x(t) = e^{\sigma t}(c_1 \sin \omega t + c_2 \cos \omega t) \pm \frac{c}{b}$$

$$x(t) = e^{-2t}(c_1 \sin 7t + c_2 \cos 7t) + \frac{15}{53}$$

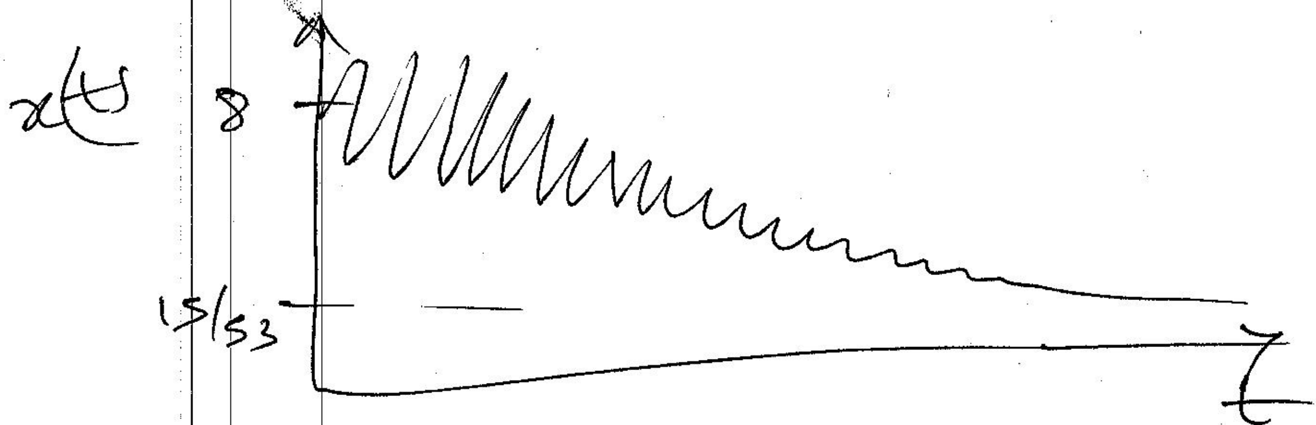
$$\dot{x}(t) = c_1\left[e^{-2t}\, 7\cos 7t + \sin 7t\,(-2)e^{-2t}\right] + c_2\left[e^{-2t}\,(-7)\sin 7t + \cos 7t\,(-2)e^{-2t}\right]$$

$x(0) = 8 = c_2 + \frac{15}{53} \qquad c_2 = 8 - \frac{15}{53} = \frac{409}{53}$

$\dot{x}(0) = -19 = 7c_1 - 2c_2$

$c_1 = -27/53$

Solution $x(t) = e^{-2t}\left(\frac{-27}{53}\sin 7t + \frac{409}{53}\cos 7t\right) + \frac{15}{53}$



Solving ODE with $\underline{\underline{\text{varying}}}$ input.

Solve $\dot{x} + 12x = 5t \qquad x(0) = 0$, Find $x(t)$

$5t$ → Not a constant
You cannot use table 2.3.2

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Take Laplace Transform on both sides of =

$$s X(s) - \cancel{x(0)}^{\,0} + 12 X(s) = \frac{5}{s^2}$$

$$s X(s) + 12 X(s) = \frac{5}{s^2}$$

$$X(s) = \frac{5}{s^2(s+12)}$$

Use Partial fractions.

$$X(s) = \frac{5}{s^2(s+12)} = \frac{c_1}{s^2} + \frac{c_2}{s} + \frac{c_3}{s+12}$$

$$c_1 = \lim_{s \to 0} s^2 X(s) = \lim_{s \to 0} \cancel{s^2} \frac{5}{\cancel{s^2}(s+12)} = \frac{5}{12}$$

$$c_2 = \lim_{s \to 0} \frac{d}{ds}\left[s^2 X(s)\right] = \lim_{s \to 0} \frac{d}{ds}\left[\frac{5}{s+12}\right]$$

$$\lim_{s \to 0} \frac{0 - 5}{(s+12)^2} = \frac{-5}{144}$$

$$c_3 = \lim_{s \to -12} \cancel{(s+12)} \left(\frac{5}{s^2\cancel{(s+12)}}\right) = \frac{5}{144}$$

$$\boxed{x(t) = \frac{5}{12} t - \frac{5}{144} + \frac{5}{144} e^{-12t}}$$

Properties of Laplace Transform

Table 2.2.2 page 42 3rd Edition of book

Prop 7 $L\left[e^{-at}x(t)\right] = X(s+a)$

Example Suppose $x(t) = t$

Find $L\left[\left(e^{-at}\right)x(t)\right]$

product of two functions of time t

Use prop 7 to solve

Find $L(x(t)) = X(s)$

Then replace s with $s+a$

for $L\left[e^{-at}x(t)\right]$

$$L\left[x(t)\right] = L\left[t\right] = \frac{1}{s^2}$$

$$L\left[e^{-at}x(t)\right] = L\left[e^{-at}t\right] = \frac{1}{(s+a)^2}$$

Ex $x(t) = \sin bt$

Find $L\left[e^{-at}x(t)\right]$

$L(\sin bt) = \frac{b}{s^2+b^2}$ | $L\left(e^{-at}\sin bt\right) = \frac{b}{(s+a)^2+b^2}$

Property 8 $L\left[t\, x(t)\right] = -\dfrac{dX(s)}{ds}$

$$x(t) = e^{-at} \qquad X(s) = \frac{1}{s+a}$$

$$L\left[t\, e^{-at}\right] = -\frac{d}{ds}\left[\frac{1}{s+a}\right]$$

$$-\left(\frac{0-1}{(s+a)^2}\right) = \frac{1}{(s+a)^2}$$