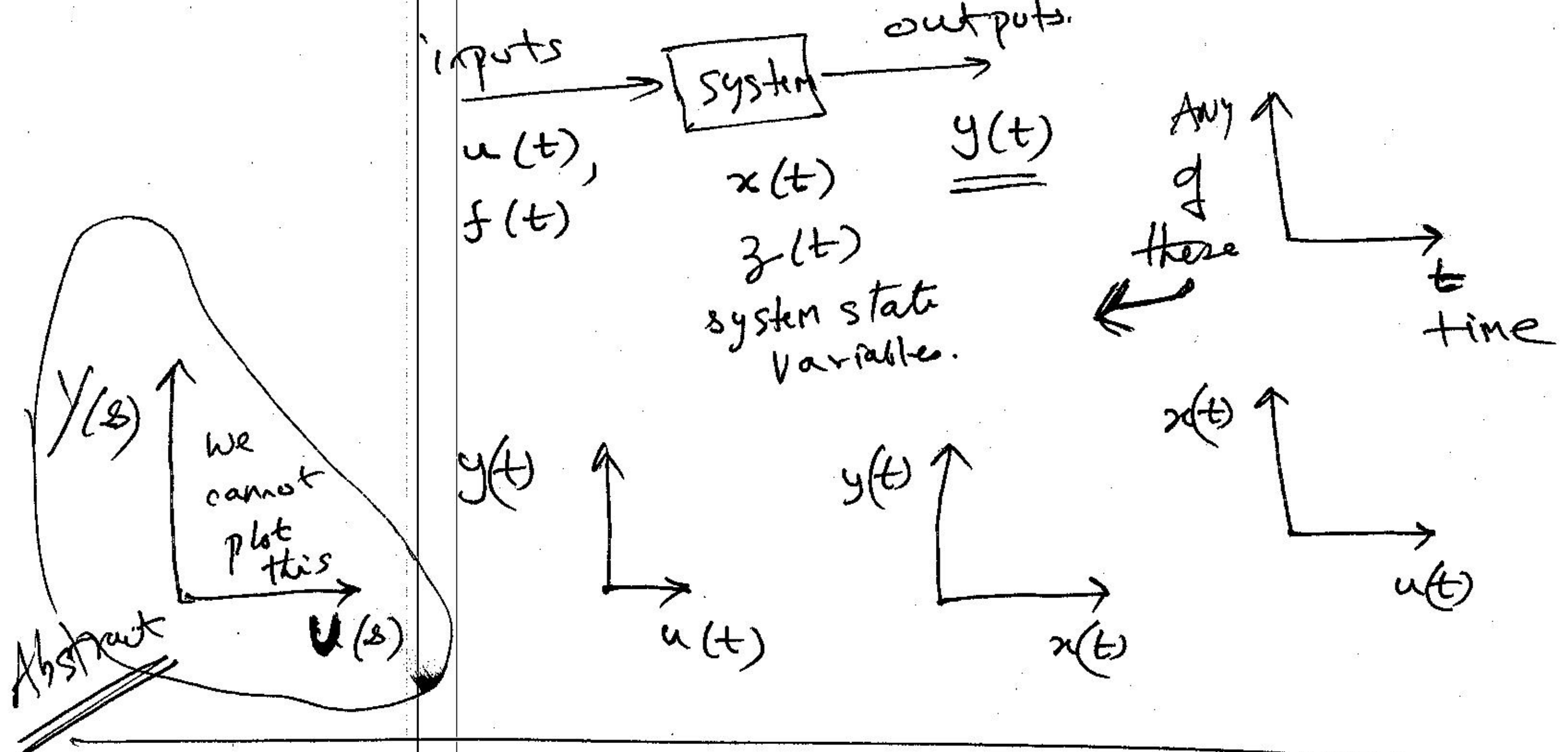
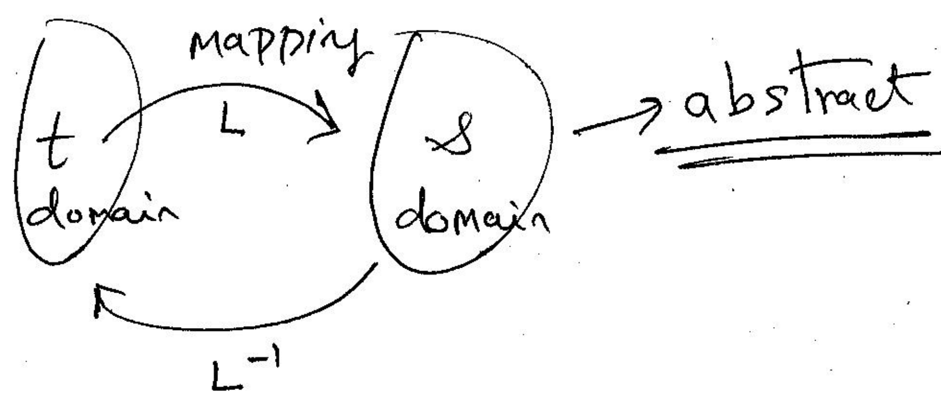
Continuous Dynamical Model (Chapter 2)

Modeled using ODE (Math 214)
(ordinary differential Equation).

Solution: You need to solve the ODE using Laplace Transforms



Laplace L
&
Inverse Laplace L^{-1}



1st order ODE

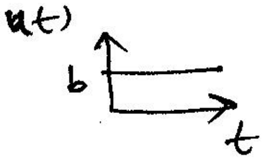
a, b are constants.

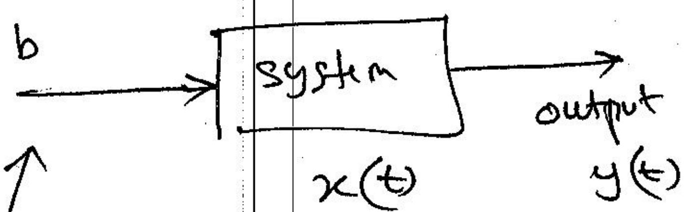
Model $\rightarrow$ $\dot{x} + ax = b$

$\dot{x} = \dfrac{dx}{dt}$

same as $\dfrac{dx}{dt} + ax(t) = b$

input to an ODE





constant input

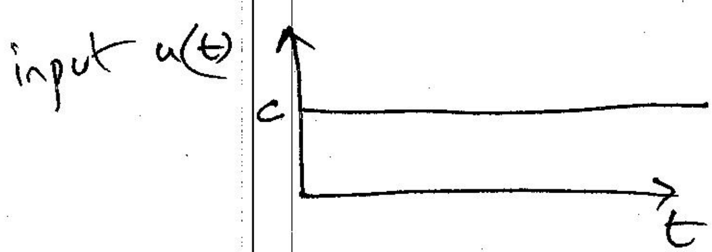
Right side of an ODE is the input to the ODE

2nd order ODE

Model $\rightarrow$ $\ddot{x} + a\dot{x} + bx = c$

$\dfrac{d^2x}{dt^2}$ $\quad \dfrac{dx}{dt}$ $\quad x(t)$ $\quad$ input to the ODE (Constant input)

a, b, c are constants



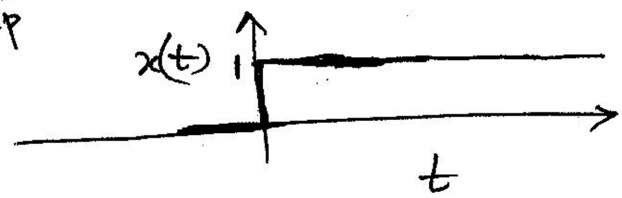
Define Laplace Transform

$$L\left(x(t)\right) = \int_0^\infty x(t)\, e^{-st}\, dt = X(s)$$

$$L^{-1}\left[X(s)\right] = x(t)$$

Example $\cancel{(t)}$ $x(t) = u_s(t)$ = unit step function $= 1$

s = step



Find $X(s)$

$$L\left[x(t)\right] = X(s) = \int_0^\infty 1 \cdot e^{-st}\, dt$$

$$= \frac{1}{-s}\left[e^{-st}\right]_0^\infty$$

$$= \frac{1}{-s}\left[0 - 1\right] = \frac{1}{s}$$

$$\boxed{L\left[x(t)\right] = L\left[u_s(t)\right] = X(s) = \frac{1}{s}}$$

$$\boxed{L[1] = \frac{1}{s}}$$

Inverse Laplace Transforms.

$$L^{-1}[X(s)] = x(t)$$

Method is called partial fractions

$$X(s) = \frac{9s+2}{s(s+8)}$$ Find $x(t)$

Step 1 : Identify the highest power of s in the denominator

step 2 : Make the coefficient of the highest power of s in the denominator $= 1$

$$\underbrace{\frac{c_1}{s} + \frac{c_2}{(s+8)}}_{\text{partial fractions}} = \frac{9s+2}{s(s+8)}$$

2 partial fractions because highest power in ~~the~~ denominator is 2

Partial fractions. Finding c_1 & c_2

Method 1

$$\frac{c_1(s+8) + s c_2}{s(s+8)} = \frac{9s+2}{s(s+8)}$$

$$c_1 s + 8c_1 + s c_2 = 9s + 2$$

Collect s terms

$$c_1 s + s c_2 = 9s$$

$$c_1 + c_2 = 9$$

Collect constant terms.

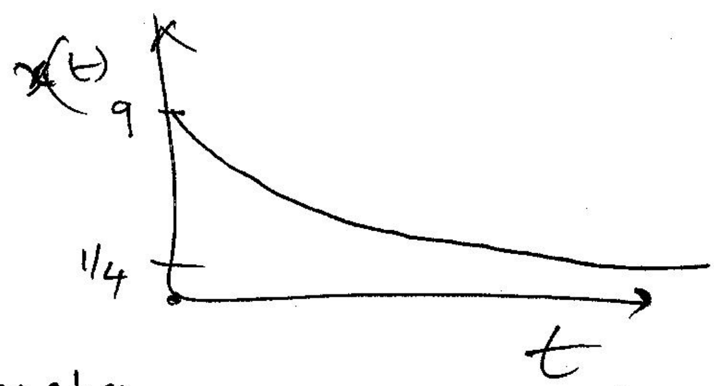
$$8c_1 = 2 \qquad c_1 = 1/4$$

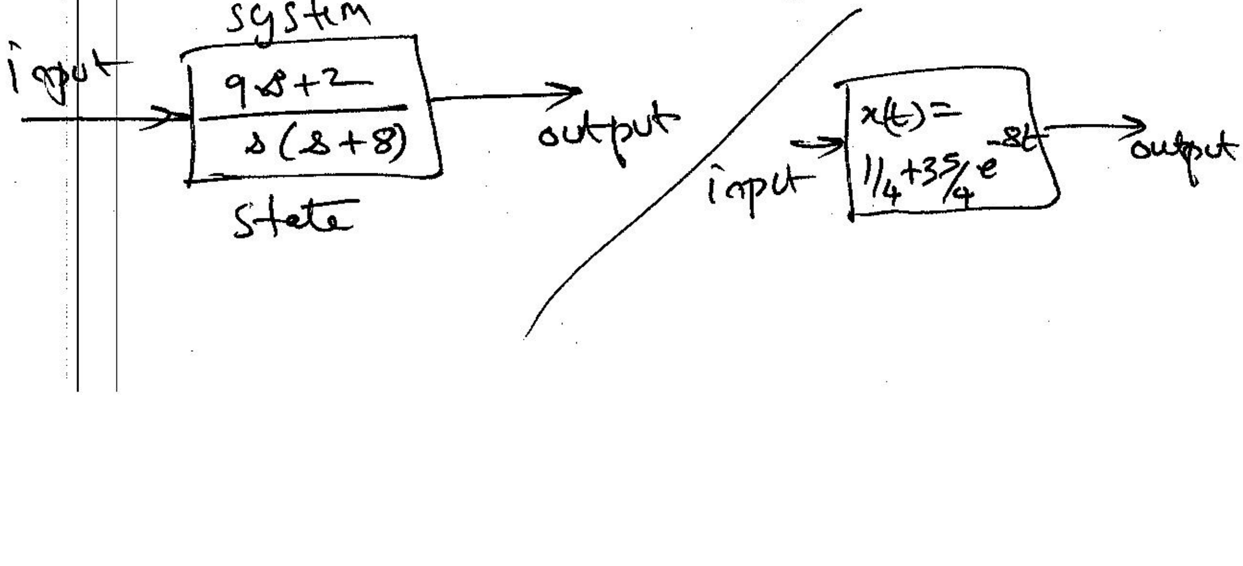
$$c_2 = 9 - 1/4 = 35/4$$

$$X(s) = \frac{1}{4}\left(\frac{1}{s}\right) + \frac{35}{4}\left(\frac{1}{s+8}\right) = \frac{9s+2}{s(s+8)}$$

$$L^{-1}(X(s)) = x(t) = \frac{1}{4} L^{-1}\left(\frac{1}{s}\right) + \frac{35}{4} L^{-1}\left(\frac{1}{s+8}\right)$$

$$\boxed{x(t) = \frac{1}{4} + \frac{35}{4} e^{-8t}}$$





Method 2 for partial fractions

$$X(s) = \frac{9s+2}{s(s+8)}$$

Partial fractions.

$$\frac{c_1}{s} + \frac{c_2}{(s+8)} = \frac{9s+2}{s(s+8)}$$

Find c_1 & c_2.

$$c_1 = \lim_{\substack{s \to 0 \\ \uparrow \\ \text{Root}}} s\,X(s) = \lim_{s \to 0} \frac{\cancel{s}(9s+2)}{\cancel{s}(s+8)} = \lim_{s \to 0} \frac{9s+2}{s+8}$$

$$= 2/8 = 1/4$$

$$c_2 = \lim_{\substack{s \to -8 \\ \uparrow \\ \text{Root}}} (s+8)\,X(s) = \lim_{s \to -8} \frac{\cancel{(s+8)}(9s+2)}{s\cancel{(s+8)}}$$

$$= \lim_{s \to -8} \frac{9s+2}{s} = \frac{-70}{-8}$$

$$= 35/4$$

Example. $X(s) = \frac{5}{s^2(3s+12)}$ Find $x(t)$

$$X(s) = \frac{5/3}{s^2(s+4)}$$

$$\frac{c_1}{s^2} + \frac{c_2}{s} + \frac{c_3}{(s+4)} = \frac{5/3}{s^2(s+4)}$$

Method 2

$$c_1 = \lim_{s \to 0} s^2 X(s) = \lim_{s \to 0} \frac{\cancel{s^2}\; 5/3}{(s+4)\,\cancel{s^2}}$$

$$\lim_{s \to 0} \frac{5/3}{4} = \boxed{5/12 = c_1}$$

$$c_2 = \lim_{s \to 0} \frac{d}{ds}\left[s^2 X(s)\right] = \lim_{s \to 0} \frac{d}{ds}\left[\frac{5/3}{s+4}\right]$$

$$\frac{d}{dt}\left[\frac{a}{b}\right] = \frac{ba' - ab'}{b^2}$$

$$\lim_{s \to 0} \left[\frac{0 - 5/3}{(s+4)^2}\right]$$

$$c_2 = \frac{-5/3}{16} = \frac{-5}{48}$$

$$c_3 = \lim_{s \to -4} (s+4) X(s) = \lim_{s \to -4} \frac{5/3\,\cancel{(s+4)}}{s^2\,\cancel{(s+4)}}$$

$$c_3 = \frac{5/3}{16} = 5/48$$

8

$$X(s) = \frac{c_1}{s^2} + \frac{c_2}{s} + \frac{c_3}{s+4}$$

$$= \frac{5}{12}\left[\frac{1}{s^2}\right] - \frac{5}{48}\left[\frac{1}{s}\right] + \frac{5}{48}\left[\frac{1}{s+4}\right]$$

L^{-1}

$$x(t) = \frac{5}{12} \times t - \frac{5}{48} \times 1 + \frac{5}{48} \times e^{-4t}$$

$$x(t) = \frac{5t}{12} - \frac{5}{48} + \frac{5}{48} e^{-4t}$$