Financial Examples

Simple Interest p.a.
per-annum.

$P \to$ Principal amount per year.

$I \to$ Rate of interest in %

$N \to$ Number of years.

$$D = \underset{\text{per year}}{\text{Simple interest}} = \frac{P \times N \times I}{100}$$

$A =$ Total amount after N years $= P + D$.

Compound Interest

$$A = P\left(1 + \left[\frac{I}{100}\right]\right)^N$$ Compounded yearly.

1000
10%
2 years
– Compounded yearly

$$S.I. = \frac{1000 \times 10 \times 2}{100} = 200$$

$$A = 1000 + 200 = \underline{\underline{1200}}$$

$$C.I. = 1000\left(1 + \left(\frac{10}{100}\right)\right)^2$$

$$= 1000\,(1.1)^2 = \underline{\underline{1210}}$$

$$A = P\left(1 + \left[\frac{I}{m \times 100}\right]\right)^n$$ Any type of compounding

n - years $m = 1$

n - ½ years (half yearly) $m = 2$

n = quarterly $m = 4$

n = monthly $m = 12$

n = weekly $m = 52$

n = days $m = 365$

Deposit \$1000 $I = 2\%$

compounded yearly $(m=1)$

How much money is paid after 25 years.

Model $\boxed{a(n+1) = r\, a(n)}$ $a(0) = 1000$

$m=1$
$n=25$

$$r = 1 + \left[\frac{I}{m \times 100}\right] = 1 + \frac{2}{100} = 1.02$$

Solution

$$a(n) = C r^n \qquad a(0) = C(1.02)^0 = 1000$$

$$C = 1000$$

$$\boxed{a(n) = 1000(1.02)^n}$$

$$a(25) = 1000(1.02)^{25} = \underline{\underline{\$1640.6}}$$

If compounded monthly $m = 12$

~~r = 1.02~~ ~~c = 1000~~

$$r = \left[1 + \left(\frac{2}{12 \times 100}\right)\right] = 1.00167$$

$a(0) = C(1.00167)^0 = 1000$

$C = 1000$

$a(n) = 1000(1.00167)^n$

after 25 years, how much is paid?

Compounded in months.

$n = 12 \times 25 = 300$ months

$a(300) = 1000(1.00167)^{300} = 1648$

Initial deposit is \$1000

$I = 2\%$

Compounded monthly.

Each year deposit \$500 starting from the 2nd year. How much in the bank after 25 years.

Model $a(n+1) = r\,a(n) + b$

$b = 500$ $\quad a(0) = 1000$

$M = 12$

$n \to$ months

$I = 2\%$

$r = 1 + \left(\frac{I}{M \times 100}\right) = 1 + \left(\frac{2}{12 \times 100}\right)$

$= 1.0016$

Solution

$a(n) = C r^n + \frac{b}{1-r}$

$a(0) = C(1.00167)^0 + \frac{(500/12)}{1 - 1.00167}$

$= 1000$

$C = 1000 - \left(\frac{500/12}{1 - 1.00167}\right)$

$c = \cancel{300401.19}\ 26950$

$n = \text{months} = 25 \times 12 = 300$ months.

$$a(300) = \cancel{300401.19}\ 26950\,(1.00167)^{300} + \frac{(500/12)}{(1-1.00167)}$$

$= \cancel{\$196165.03}\ 17{,}859$

Total Deposit $1000 + 500 \times 24 = \cancel{121000}\ 13000$

If compounded yearly

$$r = 1 + \left(\frac{I}{1 \times 100}\right) = 1.02$$

$$a(0) = c\,(1.02)^0 + \left(\frac{500}{1-1.02}\right) = 1000$$

$$c = 1000 - \left(\frac{500}{-0.02}\right) = 26000$$

$n =$ years

$$a(25) = 26000\,(1.02)^{25} + \frac{500}{(1-1.02)}$$

$= \underline{\underline{17655}}$

Yearly compounding yields less amount than monthly compounding $17655 < 17859$

Loan ~~repay~~payment

Loan :- \$1,000,000

I = 5% compounded monthly.

Loan period 30 years. (fixed interest rate).

You make monthly payments

What is your monthly payment?

$a(0) = 1{,}000{,}000$ # of months $30 \times 12 = 360$

$a(360) = 0$

$b = ?$

$$r = 1 + \left(\frac{I}{m \times 100}\right)$$

Compounded monthly. $m = 12$

$$r = 1 + \left(\frac{5}{12 \times 100}\right) = 1.00417$$

$$\boxed{\text{Model}: \quad a(n+1) = 1.00417\, a(n) + b}$$

Solution: $a(n) = c\, r^n + \left(\dfrac{b}{1-r}\right)$

$$a(0) = 1000000 = c\,(1.00417)^0 + \frac{b}{(1-1.00417)}$$

$$a(360) = 0 = c\,(1.00417)^{360} + \frac{b}{(1-1.00417)}$$

Subtract the equations.

$$1000\,000 = c\left(1-1.00417^{360}\right)$$

$$c = \frac{1000\,000}{\left(1-1.00417^{360}\right)} = \frac{1\,000\,000}{-3.473}$$

$$= -287935$$

$$1{,}000{,}000 = -287935 + \frac{b}{(1-1.00417)}$$

$$b = -5370.6$$

In 360 months you ~~pay~~ paid a total of $360 \times 5370.6 = 1{,}933{,}416$

<u>Amortization</u>

$\boxed{\text{PITI}}$

3rd Model. **Exponential**

1st
$a(n+1) = r\,a(n)$
2nd
$a(n+1) = r\,a(n) + $

Model $a(n+1) = r\,a(n) + \underbrace{\boxed{b\,s^n}}_{=b(n)}$

$r, s \rightarrow$ constants. $\rightarrow$ Not fixed (varies with n)

Solution. $a(n) = C r^n + \frac{b}{s-r} s^n$ if $r \neq s$

(OR)

$a(n) = C r^n + \frac{b n}{r} r^n$ if $r = s$

Example
Solve $a(n+1) = 2\,a(n) + 3^n$, $a(0) = 1$

$r = 2 \quad b = 1$
$s = 3$

Solution $a(n) = C\,2^n + \frac{1}{(3-2)} 3^n \rightarrow$ General solution.

Find C
$a(0) = 1 = C\,2^0 + 3^0$
$C = 0$

$\boxed{a(n) = 3^n} \rightarrow$ particular solution for $a(0) = 1$

Exponential Model for financial calculation

$I = 8\% \to$ Compounded annually $M=1$

Initial deposit $a(0) = 0$

Deposit \$100 at the end of each year.

Deposit increases 5% each year over the last year's deposit

How much money after n years?

→ Deposit Model

→ ~~Acc~~ Account balance Model.

Deposit Model

Model $b(n+1) = 1.05\, b(n)$

Solution: $b(n) = D\,(1.05)^n$

$b(0) = D\,(1.05)^0 = 100$

$D = 100$

$$\boxed{b(n) = 100\,(1.05)^n}$$

$b(0) = 100$

$b(1) = 105 = 1.05(100)$

$b(2) = 105 + \left(\frac{5}{100} \times 105\right)$

$= 1.05(105)$

$b(3) = 1.05\, b(2)$

$b(n+1) = 1.05\, b(n)$

Account balace Model

$M = 1$

$I = 8\%$

$r = \left(1 + \left(\frac{I}{M \times 100}\right)\right)$

$r = \left(1 + \frac{8}{100}\right) = 1.08$

$a(0) = 0$

$a(n+1) = 1.08\, a(n) + b(n)$

Model for account balance $\boxed{a(n+1) = 1.08\, a(n) + 100(1.05)^n}$

$r = 1.08 \quad s = 1.05$

$b = 100$

<u>Solution</u> $a(n) = C(1.08)^n + \dfrac{100\,(1.05)^n}{(1.05 - 1.08)}$

Solve for C $\quad a(0) = 0$

$$a(0) = 0 = C + \frac{100}{(-0.03)}$$

$$C = \frac{100}{0.03}$$

Solution: $a(n) = 3333.33\,(1.08)^n - 3333.33\,(1.05)^n$

$$\boxed{a(n) = 3333.33\left[(1.08)^n - (1.05)^n\right]}$$

4th Model is polynomial

Model $a(n+1) = r\,a(n) + \text{polynomial}$.

order of a polynomial.

$n \to$ 1st order — linear polynomial.

$n^2 \to$ 2nd order $\to$ quadratic equations.

n^3 — 3rd order — cubic equation

Solution.

$r \neq 1$		Add
	1st order polynomial	$An+B$
	2nd order "	An^2+Bn+D
	3rd order "	An^3+Bn^2+Dn+E

Solution: $a(n) = c\,r^n + (\quad)$

Ex

M: $a(n+1) = 2a(n) + n$

S: ~~[illegible]~~ $a(n) = C\,2^n + An+B$

M: $a(n+1) = 2a(n) + n+1$

S: $a(n) = C\,2^n + An+B$

M: $a(n+1) = 2a(n) + n^2$

S: $a(n) = C\,2^n + An^2+Bn+D$

M: $a(n+1) = 2a(n) + n^2+4n$

S: $a(n) = C\,2^n + An^2+Bn+D$

$$r = 1 \begin{cases} 1^{st} \text{ order} \\ 2^{nd} \text{ order} \\ 3^{rd} \text{ order} \end{cases}$$

Add

$n(An+B)$

$n(An^2+Bn+D)$

$n(An^3+Bn^2+Dn+E$

Example $\underline{M}$ $a(n+1) = 2a(n)+n+1$

$a(0)=0$

Solve.

$r \neq 1$

Generic Solution : $a(n) = C\,2^n + An+B$.

$a(n+1) = C\,2^{n+1} + A(n+1)+B$

$C\,2^{n+1} + An+A+B = 2(C2^n)+2An+2B+n+1$

$\cancel{C\,2^{n+1}} + An + A + B = \cancel{C2^{n+1}} + 2An + 2B + n + 1$

Collect n terms $\rightarrow$ $A\cancel{n} = 2A\cancel{n} + \cancel{n}$

Collect constants $\rightarrow$ $A+B = 2B+1$

$A = 2A+1 \Rightarrow \boxed{A = -1}$

$A = B+1 \Rightarrow B = A-1 = -1-1$

$\boxed{B = -2}$

Generic Solution: $a(n) = C\,2^n - n - 2$

Find C $\quad a(0)=0 = C\,2^0 - 0 - 2$

$0 = C - 2 \quad \boxed{C = 2}$

Particular solution for $a(0)=0$: $a(n) = (2)2^n - n - 2$

$\boxed{a(n) = 2^{n+1} - n - 2}$

2nd Example

Model $a(n+1) = a(n) + n + 1$ $a(0) = 0$

Solve it $r = 1$

Generic Solution $a(n) = c\, r^n + n(An + B)$

$$a(n+1) = c\, r^{n+1} + (n+1)(A(n+1)+B)$$

$$\cancel{c} + (n+1)(An + A + B) = \cancel{c} + An^2 + nB + n + 1$$

$$\cancel{An^2} + An + \cancel{nB} + An + A + B = \cancel{An^2} + \cancel{nB} + n + 1$$

$$2An + A + B = n + 1$$

Collect n terms $\Rightarrow$ $2A\cancel{n} = \cancel{n}$

Collect constants $\Rightarrow$ $A + B = 1$

$$2A = 1 \quad \boxed{A = 1/2}$$

$$\boxed{B = 1/2}$$

Generic solution: $a(n) = c + n(1/2\, n + 1/2)$

$$a(0) = 0 = c + 0(1/2 \times 0 + 1/2)$$

$$c = 0$$

Particular solution for $a(0) = 0$ $a(n) = n\left(\frac{1}{2} n + 1/2\right)$

$$a(n) = \frac{1}{2} n (n+1)$$

why is it called Difference Equation?

1st order difference equation

$$a(n+1) = r\,a(n) + b.$$

+ b=0

+ Expo ($b s^n$)

+ polynomial.

$$\underbrace{a(n+1) - r\,a(n)}_{\text{Difference.}} = 0 \text{ or } b$$

or Expo or poly.

5th model

2nd order Difference Equation.

Model $a(n+2) = b_1 a(n+1) + b_2 a(n)$

2 Step dependency

Solution: $a(n) = C_1 r^n + C_2 s^n$ if $r \neq s$

(OR)

$a(n) = C_1 r^n + n C_2 r^n$ if $r = s$

r, s are called characteristic roots.

$a(n+2) \Rightarrow x^2$

$a(n+1) \Rightarrow x$

$a(n) \Rightarrow 1$

characteristic equation

$$\boxed{x^2 = b_1 x + b_2}$$

Find roots r and s

Ex

Model $a(n+2) = -3.5\, a(n+1) + 2\,(a(n))$

$a(0) = 3$
$a(1) = -3$

$b_1 = -3.5 \quad b_2 = 2$

$x^2 = -3.5x + 2$ → characteristic equation

$$\boxed{x^2 + 3.5x - 2 = 0}$$

Quadratic Roots

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Find the roots.

Roots are $0.5 = r$
and $-4 = s$

Generic Solution $a(n) = c_1 r^n + c_2 s^n$ if $r \neq s$

↳ $a(n) = c_1 (0.5)^n + c_2 (-4)^n$

Need 2 intial conditions:

$a(0) = 3 = c_1 (0.5)^0 + c_2 (-4)^0$
$a(1) = -3 = c_1 (0.5)^1 + c_2 (-4)^1$

$3 = c_1 + c_2$

$-3 = 0.5c_1 - 4c_2$

Solve these equations to find c_1 & c_2

$c_1 = 2 \quad c_2 = 1$

Particular solution for $a(0) = 3$, $a(1) = -3$ is $\boxed{a(n) = 2(0.5)^n + (-4)^n}$