Discrete Dynamical Models

Not in your book

Use Difference Equation

You take 20 ml of a drug :- Kidneys clear 20% of the drug every 4 hours.

1) Build a model for this system
2) How much drug left after 16 hrs?

$t=0$ $a(0)=20$

Time — quantity of drug — Time index

$a(n)$

$a[0]=20$

$t=0$

$t=4$ hrs $a(1)=20-\left[\frac{20}{100}\times 20\right]$

$a(1)=20[1-0.2]$

$=0.8\times 20 = 16$

time index → $n=0$ $n=1$ $n=2$ →

Real/actual time → $t=0$ $t=4$hrs $t=8$hrs →

$$\boxed{a(1)=0.8a(0)}$$

$t=8$hrs $a(2)=16-\left[\frac{20}{100}\times 16\right]=16(1-0.2)$

$=0.8\times 16$

$=12.8$

$$\boxed{a(2)=0.8\,a(1)}$$

$t=12$ hrs $a(3)=0.8\,a(2)=0.8\times 12.8=10.24$

$t=16$ hrs $a(4)=0.8\,a(3)=10.24\times .8=8.1$

$$a(n+1) = r\,a(n)$$ Model.

↓ constant

If $r = -1$, $a(n+1) = -a(n)$

$a(0) = 20$

$a(1) = -20$

$a(2) = 20$

$a(3) = -20$

Solution to the Model

$$a(n) = C r^n$$ General Solution.

To prove substitute the solution into the model.

$$a(n+1) = C r^{(n+1)}$$

$$C r^{(n+1)} = r\,C r^n = C r^{(n+1)}$$

Kidney Example

If $a(0) = 20$ then find C

Model $a(n+1) = 0.8\,a(n)$, $r = 0.8$

$$a(0) = C(0.8)^0 = 20$$

$C = 20$

initial condition

$$a(n) = 20(0.8)^n$$ Particular solution

Particular to the given initial condition $a(0) = 20$

First model

M→ $a(n+1) = r\,a(n)$

S→ $a(n) = C r^n$

Find C using initial condition

How much after 100 hrs

$t = 100$ $\quad n = \frac{100}{4} = 25$

$a(25) = 20(0.8)^{25} =$

2nd ~~Homo~~ model

Model $a(n+1) = r\,a(n) + b \qquad b \neq 0$

(r: constant; b: constant)

Solution $a(n) = C r^n + \frac{b}{1-r} \qquad r \neq 1$

OR

$= bn + C \qquad r = 1$

Example. $a(n+1) = 0.5\,a(n) + 2$, $a(0) = 3$ (initial condition)

Solve. $r = 0.5$

$b = 2$

Solution. $a(n) = C(0.5)^n + \frac{2}{1-0.5}$

(General Soln) $a(n) = C(0.5)^n + 4$

$a(0) = C(0.5)^0 + 4 = 3$

Particular Solution. $C + 4 = 3$

$C = -1$

Particular Solution $a(n) = -1(0.5)^n + 4$

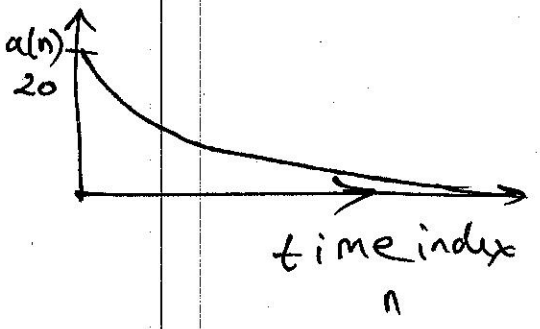
Stability

Is $a(n)$ stable?

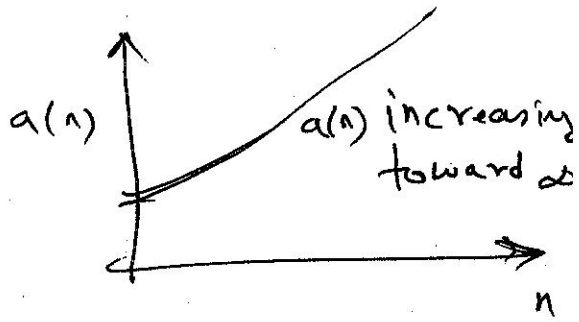
as $n \to \infty$

① $|r| < 1$ stable

$|r|<1$



② $|r| > 1$ unstable



③ $r = 1$ → stable (constant)

④ $r = -1$ → neutrally stable (Toggles between 2 points) oscillates.

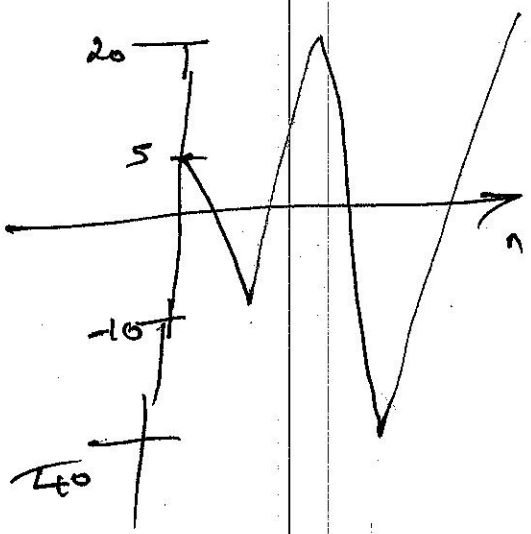
Example $a(n+1) = -2\,a(n)$

$a(0) = 5$

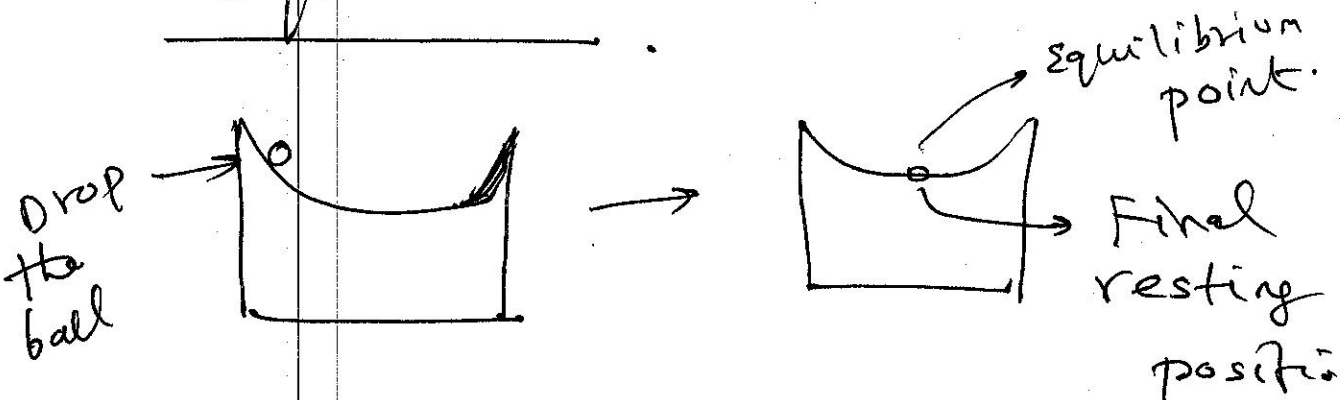
$a(1) = -10$

$a(2) = 20$

$a(3) = -40$



Equilibrium



$a(n+1) = r\,a(n) + b$

Find equilibrium point.

At equilibrium $a(n+1) = a(n) = y$

$$y = ry + b \qquad y = \frac{b}{1-r}$$

$a(n+1) = 0.5\,a(n) + 2$

what is the equilibrium point

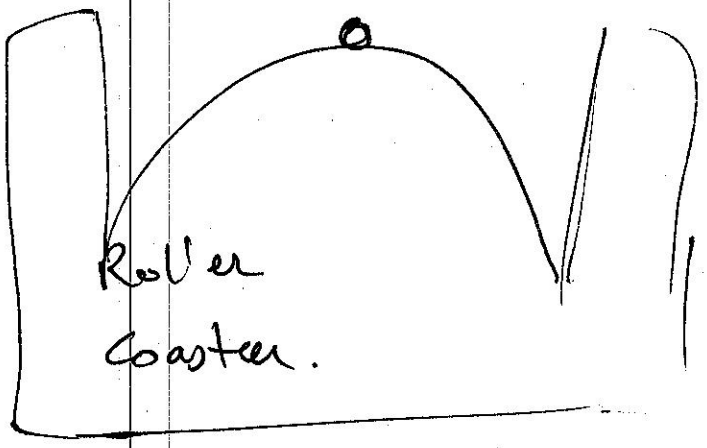
$$y = \frac{b}{1-r} = \frac{2}{(1-0.5)} = 4$$

$a(0) = 4$
$a(1) = 4$
$a(2) = 4$

$a(0) = 3$
$a(1) = 3.5$
$a(2) = 3.75$
↓
4

$a(0) = 5$
$a(1) = 4.5$
$a(2) = 4.25$
↓
4

Do unstable systems have equilibrium point?



Yes

$$a(n+1) = +5a(n) + 10.$$

$$r = 5 \quad b = 10.$$

$$Y = 5Y + 10 \qquad Y = \frac{10}{-4} = -2.5$$

$a(0) = -2.5$
$a(1) = -2.5$
$a(2) = -2.5$

$a(0) = 2$
$a(1) = 20$
$a(2) = 110$

$a(0) = -5$
$a(1) = -15$
$a(2) = -65$

If you start away from equilibrium you can never reach equilibrium.

If you are at equilibrium then you will always remain there.