A Sampling Theory for Compact Sets in Euclidean Space

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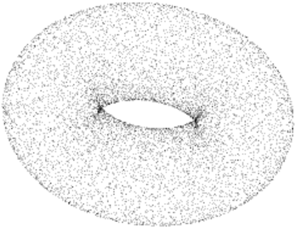
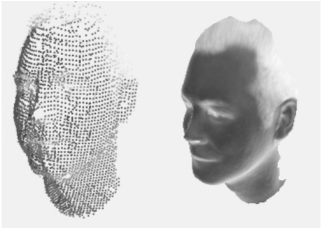
Motivation

We can create point clouds via sampling. An important question then concerns finding a condition which guarantees the original object can be reconstructed accurately.





Definition

The **distance function** R_K of a compact set K of $\mathbb{R}^n$ associates to each point $x \in \mathbb{R}^n$ its distance to K,

$$R_K(x) = \min_{y \in K} d(x, y)$$

Note that R_K completely characterizes K since $K = \{x \in \mathbb{R}^n | R_K(x) = 0\}$.

Definition

For a positive number α, define the $\alpha-$**offset** of K, denoted K^α to be the set

$$K^\alpha = \{x \in \mathbb{R}^n | R_K(x) \leq \alpha\}$$

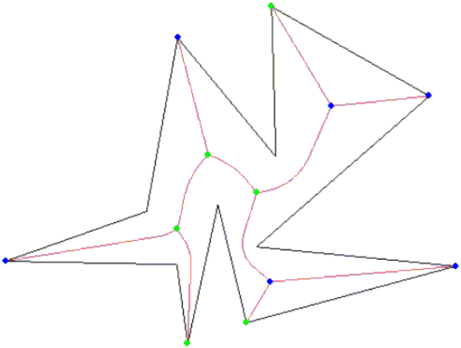
Definition

The **Hausdorff distance** $d_H(K, K')$ between two compact sets $K, K' \subset \mathbb{R}^n$ is the minimum α for which $K \subset (K')^\alpha$ and $K' \subset K^\alpha$. Note that this is equivalent to

$$d_H(K, K') = \sup_{x \in \mathbb{R}^n} |R_K(x) - R_{K'}(x)|$$

Definition

The **Medial Axis** of an compact set K is the set of all points having more than one closest point on the boundary of K.



Gradient and Flow of Distance Functions

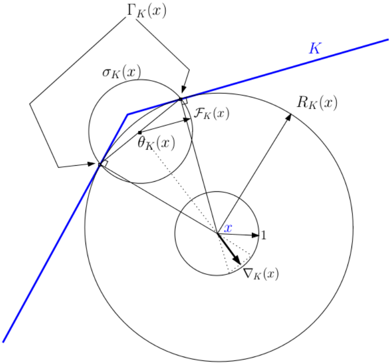
Given a compact set K of $\mathbb{R}^n$, its associated distance function R_K is not differentiable on the medial axis of $\mathbb{R}^n \backslash K$.

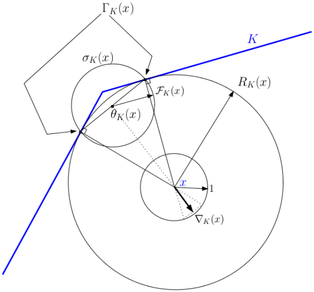
To overcome this issue, we seek a *Generalized Gradient* function $\nabla_K : \mathbb{R}^n \to \mathbb{R}^n$ which agrees with the usual gradient of R_K at points where R_K is differentiable.

Generalized Gradient Flow

Flurry of definitions:

$\Gamma_K(x)$: set of points in K closest to x (i.e. $\{y \in K | d(x,y) = R_K(x)\}$)
$\sigma_K(x)$: unique smallest ball enclosing $\Gamma_K(x)$.
$\Theta_K(x)$: center of $\sigma_K(x)$.
$\mathcal{F}_K(x)$: radius of $\sigma_K(x)$.





Definition

The **generalized gradient flow** ∇_K of R_K is defined as

$$\nabla_K(x) = \frac{x - \Theta_K(x)}{R_K(x)}$$

We define $\nabla_K(x) = 0$ for all $x \in K$. Note that $\|\nabla_K(x)\| \leq 1$ for all $x \in \mathbb{R}^n$.

A few technical points

Although ∇_K is not continuous, it can be shown that Euler shcemes using ∇_K converge uniformly toward a continuous flow $\mathcal{C} : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$.

Parameterizing an integral line of this flow by arc length, we get a map $s \mapsto \mathcal{C}(t(s), x)$ and so we can express the value of R_K at the point $\mathcal{C}(t(\ell), x)$ by integration along the curve with length ℓ,

$$R_K(\mathcal{C}(t(\ell), x)) = R_K(x) + \int_0^\ell \|\nabla_K(\mathcal{C}(t(s), x)\| ds$$

It can be shown that $\mathcal{F}_K$ and R_K increase along trajectories of the flow.

Critical Points for Distance Functions

Definition

A point $x \in K$ is a **critical point** of R_K if $\nabla_K(x) = 0$.

The topology of K^α are closely related to the critical *values* of R_K.

Definition

The **weak feature size** of K, denoted $\mathrm{wfs}(K)$, is the infimum of the positive critical values of R_K. Equivalently, it is the minimum distance between K and the set of critical points of R_K.

The next lemma shows wfs may be viewed as the "minimum size of the topological features" of the set K.

Lemma

If $0 < \alpha, \beta < \mathrm{wfs}(K)$, then K^α and K^β are homeomorphic and even isotopic. The same holds for the complements of K^α and K^β.

Isotopic: roughly speaking, two subspaces of $\mathbb{R}^n$ are isotopic if they can be deformed one into each other without tearing or self-intersection.

Theorem

Let K and K' be compact subsets of $\mathbb{R}^n$ and ϵ such that $\mathrm{wfs}(K) > 2\epsilon$, $\mathrm{wfs}(K') > 2\epsilon$, and $d_H(K, K') < \epsilon$. Then

- (i) $\mathbb{R}^n \backslash K$ and $\mathbb{R}^n \backslash K'$ have the same homotopy type.
- (ii) If $0 < \alpha \leq 2\epsilon$ then K^α and $(K')^\alpha$ have the same homotopy type.

We can generalize the notion of critical point.

Definition

A μ-**critical point** of the compact set K is a point x such that $\|\nabla_K(x)\| \leq \mu$.

μ critical points exhibit some stability.

Critical Point Stability Theorem

Let K and K' be two compact subsets of $\mathbb{R}^n$ and $d_H(K, K') \leq \epsilon$. For any μ-critical point x of K, there is a $(2\sqrt{\epsilon / R_K(x)} + \mu)$-critical point of K' a distance of at most $2\sqrt{\epsilon R_K(x)}$ from x.

Definition

Given a compact set $K \subset \mathbb{R}^n$, its critical function $\chi_K : (0, +\infty) \to \mathbb{R}_+$ is given by

$$\chi_K(d) = \min_{x \in R_K^{-1}(d)} \|\nabla_K(x)\|$$

Note that $R_K^{-1}(d)$ are all the points which are a distance d from K and so $\chi_K(d)$ is the minimum norm of the gradient at these points.

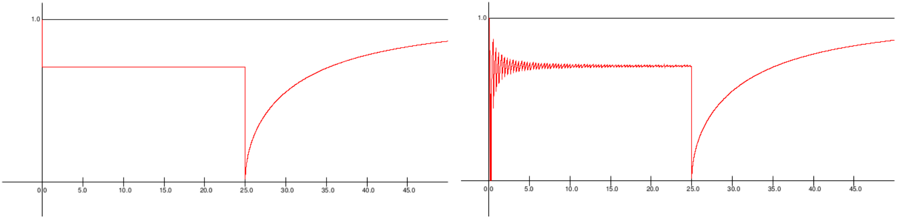
Like μ-critical points, the critical function also has stability.

Critical Function Stability Theorem

Let K and K' be two compact subsets of $\mathbb{R}^n$ and $d_H(K, K') \leq \epsilon$. For all $d \geq 0$, we have:

$$\inf\{\chi_{K'}(u) | u \in I(d, \epsilon)\} \leq \chi_K(d) + 2\sqrt{\frac{\epsilon}{d}}$$

where $I(d, \epsilon) = [d - \epsilon, d + 2\chi_K(d)\sqrt{\epsilon d} + 3\epsilon]$.



Critical function of a square with side length 50 in $\mathbb{R}^3$ (left) and the critical function of a sampling of the square (right).

Defintion

The μ-**reach** $r_\mu(K)$ of a compact set $K \subset \mathbb{R}^n$ is defined as

$$r_\mu(K) = \inf\{d | \chi_K(d) < \mu\}$$

and one last definition...

Definition

Given two non-negative real numbers κ and μ, we say that a compact set $K \subset \mathbb{R}^n$ is a (κ, μ)-**approximation** of a compact set $K' \subset \mathbb{R}^n$ if the Hausdorff distance between K and K' does not exceed κ times the μ-reach of K'.

Reconstruction Theorem

Let $K \subset \mathbb{R}^n$ be a (κ, μ)-approximation of a compact set K'. Let α be such that

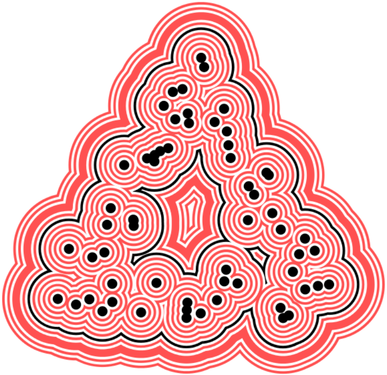
$$\frac{4d_H(K, K')}{\mu^2} \leq \alpha < r_\mu(K') - 3d_H(K, K')$$

If

$$\kappa < \frac{\mu^2}{5\mu^2 + 12}$$

then the complement of K^α is homotopy equivalent to the complement of K', and K^α is homotopy equivalent to $(K')^\eta$ for sufficiently small η.

Example



The distance function to a sampling of an equilateral triangle. If the offset parameter is appropriately chosen, then the offset of the sampling (tis boundary is shown in bold), is homotopy equivalent to the triangle.

Future Work

- Do the sampling conditions allow for the recovery of differential information?
- This approach assumes the magnitude of ther perturbation is uniform over the object, since we use Hausdorff distance. Can the ideas be generalized to design a non-uniform sampling theory?
- What if the ambient metric space is non-Euclidean?

References

F. Chazal, D. Cohen-Steiner, A. Lieutier (2006)
A Sampling Theory for Compact Sets in Euclidean Space
Proceedings of the Symposium on Computational Geometry