Multiplication Operators on the Lipschitz Space of a Tree

Part II: Spectrum

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This presentation will continue to study the multiplication operator M_ψ given in [1]. Today, we will look at the spectrum of this operator.

We start with a review from last week.

Recall

Let X be a complex Banach space consisting of functions defined on a set Ω. Let $\psi : \Omega \to \mathbb{C}$.

- The *multiplication operator with symbol* ψ is defined as the operator $M_\psi : X \to X$ given by $M_\psi(f) = \psi f$.

We will be investigating the following case:

- Ω is an infinite tree.
- We consider Ω as a metric space with distance given by the length of the unique path between two vertices.
- X is the space of complex-valued Lipschitz functions on T; i.e. complex-valued functions f for which there exists a constant C such that, for all $u, v \in \Omega$,

$$|f(v) - f(u)| \leq Cd(u, v)$$

We denote this space, $\mathcal{L}$.

Recall

Recall the following definitions:

- The norm of elements in $\mathcal{L}$: For $f \in \mathcal{L}$,

$$\|f\| = |f(o)| + \sup_{0 \neq v \in T} |f(v) - f(v^-)|$$

- Derivative at a vertex: For $v \in T$,

$$Df(v) = |f(v) - f(v^-)|$$

- The little Lipschitz space $\mathcal{L}_0$ is the subspace of $\mathcal{L}$ consisting of all functions f on T such that

$$\lim_{|v| \to \infty} Df(v) = 0$$

Spectrum

Definition

Let A be a bounded operator on a Banach space X. The **spectrum** of A, denoted $\sigma(A)$, is defined as

$$\sigma(A) = \{\lambda \in \mathbb{C} | A - \lambda I \text{ is not invertible}\}$$

where I is the identity operator.

Note that if $(A - \lambda I)^{-1}$ exists, it must be linear since $(A - \lambda I)$ is linear. Also, if $A - \lambda I$ is a bijection, by the bounded inverse theorem, $(A - \lambda I)^{-1}$ would be bounded.

Point Spectrum

It is therefore useful to look at when $A - \lambda I$ will not be injective:

Definition

Let A be a bounded linear operator on a Banach space X. The set of eigenvalues, $\sigma_p(A)$, is called the **point spectrum** of A. Recall,

$$\sigma_p(A) = \{\lambda \in \mathbb{C} |\, \ker(A - \lambda I) \neq 0\}$$

i.e. $A - \lambda I$ is not injective.

Approximate Point Spectrum

Definition

Let A be a bounded linear operator on a Banach space X. The **approximate point spectrum** $\sigma_{\mathrm{ap}}(A)$ of A consists of all $\lambda \in \mathbb{C}$ for which there exists a sequence of norm 1 vectors $\{x_n\}$ such that

$$\|(A - \lambda I)x_n\| \to 0 \text{ as } n \to \infty$$

i.e. Ax_n and λx_n converge to each other.

By this definition, $\sigma_p(A) \subseteq \sigma_{\mathrm{ap}}(A)$ and so

$$\sigma_p(A) \subseteq \sigma_{\mathrm{ap}}(A) \subseteq \sigma(A)$$

Spectrum Classification: Lemma I

The next two lemmas will be used to prove that

$$\partial\sigma(A) \subseteq \sigma_{\mathrm{ap}}(A)$$

These results are from [2]. I will give the first lemma without proof.

Lemma

Let A be an operator on a Banach Space X. If there exists A_0, B_0 such that $B_0 A_0 = I$ and $\|A - A_0\| < \|B_0\|^{-1}$, then A is left invertible.

Spectrum Classification: Lemma II

Lemma

If $\lambda \in \partial\sigma(A)$ and $\{\lambda_n\}$ is a sequence in $\mathbb{C}\backslash\sigma(A)$ such that $\lambda_n \to \lambda$, then $\|(A - \lambda_n I)^{-1}\| \to \infty$ as $n \to \infty$.

Proof

Suppose otherwise. Then there exists a constant M and a subsequence $\{\lambda_{n_k}\}$ such that $\|(A - \lambda_{n_k} I)^{-1}\| \leq M$ for all k. Since $\lambda_{n_k} \to \lambda$, we can pick k large enough so that $|\lambda_{n_k} - \lambda| < M^{-1}$. Then $\|(\lambda_{n_k} - \lambda)I\| = \|(A - \lambda I) - (A - \lambda_{n_k} I)\| < \|(A - \lambda_{n_k} I)^{-1}\|^{-1}$ which, by the above lemma, would imply $(A - \lambda I)$ is invertible. This is a contradiction since $\lambda \in \sigma(A)$ and therefore we have the desired result.

Spectrum Classification: Theorem

Theorem

Let A be a bounded linear operator on a Banach space X. Then

$$\partial\sigma(A) \subseteq \sigma_{\mathrm{ap}}(A)$$

Proof

Let $\{\lambda_n\} \to \lambda$ as we had before. Recall,

$$\|(A-\lambda_n)^{-1}\| = \sup_{\|x\|=1} \|(A-\lambda_n)^{-1}(x)\|$$

Now, let $\{x_n\}$ be a sequence in X such that $\|x_n\| = 1$ and $\|(A-\lambda_n)^{-1}x_n\| > \|(A-\lambda_n)^{-1}\| - n^{-1}$.

Spectrum Classification: Proof of Theorem

Let $\alpha_n = \|(A - \lambda_n)^{-1} x_n\|$. Then by the previous lemma, $\alpha_n \to \infty$. Now set $y_n = \alpha_n^{-1}(A - \lambda_n)^{-1} x_n$ (obviously $\|y_n\| = 1$). Then we have

$$\begin{aligned}(A - \lambda)y_n &= (A - \lambda_n)y_n + (\lambda - \lambda_n)y_n \\ &= \alpha_n^{-1} x_n + (\lambda - \lambda_n)y_n\end{aligned}$$

This means $\|(A - \lambda)y_n\| \le \alpha_n^{-1} + |\lambda - \lambda_n|$ and so $\|(A - \lambda)y_n\| \to 0$ as $n \to \infty$. Recall that this was the definition of an element of the approximate point spectrum. Therefore, $\lambda \in \sigma_{\mathrm{ap}} A$.

Recall a Theorem

For a function ψ on T, define

$$\sigma_\psi = \sup_{v \neq 0} |v| D\psi(v)$$

Note that if $\sigma_\psi < \infty$, then $\psi \in \mathcal{L}_0$.

Theorem

Let T be a tree and ψ a function on T, then the following are equivalent,

(a) *M_ψ is bounded on $\mathcal{L}$.*

(b) *M_ψ is bounded on $\mathcal{L}_0$.*

(c) *$\psi \in L^\infty$ and σ_ψ is finite.*

Theorem 5.1

Theorem

Let M_ψ be a bounded multiplication operator on $\mathcal{L}$ or $\mathcal{L}_0$. Then

(a) $\sigma_p(M_\psi) = \psi(T)$;

(b) $\sigma(M_\psi) = \sigma_{ap}(M_\psi) = \overline{\psi(T)}$

Proof of Theorem 5.1a

Proof

Since boundedness of a multiplication operator on $\mathcal{L}$ implies boundedness on $\mathcal{L}_0$, we will only consider the former.

$$\sigma_p(M_\psi) = \psi(T)$$

Let $\lambda \in \sigma_p(M_\psi)$. Then there exists $0 \not\equiv f \in \mathcal{L}$ such that $\psi f = \lambda f$. This means there exists a vertex v such that $f(v) \neq 0$ and $\psi(v)f(v) = \lambda f(v)$. Therefore, $\psi(v) = \lambda$ and $\lambda \in \psi(T)$.

Now let $\lambda \in \psi(T)$. Then there exists $v \in T$ such that $\psi(v) = \lambda$. Thus since $\chi_v \not\equiv 0$ but $(M_\psi - \lambda I)\chi_v \equiv 0$, we see $(M_\psi - \lambda I)$ is not injective and therefore, $\lambda \in \sigma_p(M_\psi)$.

Proof of Theorem 5.1b

$$\sigma(M_\psi) = \overline{\psi(T)}$$

One direction is obvious. By part (a) and the fact that $\sigma(M_\psi)$ is closed, we have $\overline{\psi(T)} \subseteq \sigma(M_\psi)$.

Proof of Theorem 5.1b

Next, let $\lambda \notin \overline{\psi(T)}$. This means there exists $c > 0$ such that $|\psi(v) - \lambda| \geq c$ for all $v \in T$. Therefore, the function $\varphi_\lambda : T \to \mathbb{C}$ defined by $\varphi_\lambda(v) = (\psi(v) - \lambda)^{-1}$ is bounded on T. Also,

$$\begin{aligned}
\sigma_{\varphi_\lambda} = \sup_{v \neq o} |v| D_{\varphi_\lambda}(v) &= \sup_{v \neq 0} |v| |\varphi_\lambda(v) - \varphi_\lambda(v^-)| \\
&= \sup_{v \neq 0} |v| |(\psi(v) - \lambda)^{-1} - (\psi(v^-) - \lambda)^{-1}| \\
&= \sup_{v \neq 0} |v| \left| \frac{\psi(v) - \psi(v^-)}{(\psi(v) - \lambda)(\psi(v^-) - \lambda)} \right| \\
&\leq \sup_{v \neq 0} \frac{|v|}{c^2} D_\psi(v) \\
&\leq \frac{1}{c^2} \sigma_\psi
\end{aligned}$$

Proof of Theorem 5.1b

Then, using a previous theorem and the fact that $\sigma_{\varphi_\lambda} \leq \frac{1}{c^2}\sigma_\psi < \infty$, we have M_{φ_λ} is bounded on $\mathcal{L}$. Therefore, $M_{\psi-\lambda} = M_\psi - \lambda I$ is invertible on $\mathcal{L}$ which means $\lambda \notin \sigma(M_\psi)$. Hence, $\sigma(M_\psi) = \overline{\psi(T)}$.

Finally, since $\partial\sigma(M_\psi) \subseteq \sigma_{\mathrm{ap}}(M_\psi) \subseteq \sigma(M_\psi)$ and by part (a), we have $\sigma_{\mathrm{ap}}(M_\psi) = \sigma(M_\psi) = \overline{\psi(T)}$ as desired.

Boundedness from Below

Definition

Let A be a bounded operator on a Banach space X is *bounded below* if there exists $C > 0$ such that $\|Ax\| \geq C\|x\|$ for all $x \in X$.

Boundedness from Below

The following theorem is from [2].

Theorem

For a bounded operator A on a Banach space X and for $\lambda \in \mathbb{C}$, the following are equivalent,

(a) $\lambda \notin \sigma_{ap}(A)$

(b) $A - \lambda I$ *is injective and has closed range*

(c) $A - \lambda I$ *is bounded below.*

Boundedness from Below

From theorem 5.1 and the previous theorem, we conclude that if M_ψ is bounded on $\mathcal{L}$ or $\mathcal{L}_0$, then M_ψ is bounded below $\iff$ $0 \notin \overline{\psi(t)}$. Thus,

Theorem

The bounded operator M_ψ on $\mathcal{L}$ or $\mathcal{L}_0$ is bounded below if and only if

$$\inf\{|\psi(v)| : v \in T\} > 0$$

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