## Reality Check 1 Kinematics of the Stewart platform

A Stewart platform consists of six variable length struts, or prismatic joints, supporting a payload. Prismatic joints operate by changing the length of the strut, usually pneumatically or hydraulically. As a six-degree-of-freedom robot, the Stewart platform can be placed at any point and inclination in three-dimensional space that is within its reach.

To simplify matters, the project concerns a two-dimensional version of the Stewart platform. It will model a manipulator composed of a triangular platform in a fixed plane controlled by three struts, as shown in Figure 1.14. The inner triangle represents the planar Stewart platform whose dimensions are defined by the three lengths $L_1$, $L_2$, and $L_3$. Let $\gamma$ denote the angle across from side $L_1$. The position of the platform is controlled by the three numbers $p_1$, $p_2$, and $p_3$, the variable lengths of the three struts.

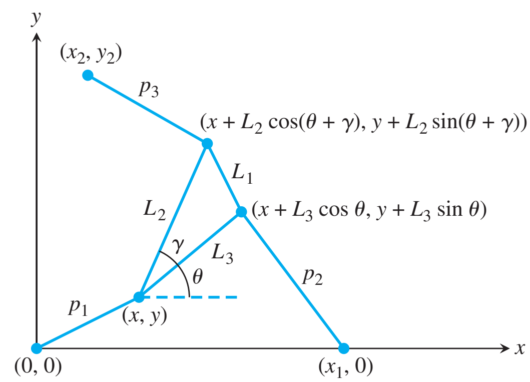


**Figure 1.14 Schematic of planar Stewart platform.** The forward kinematics problem is to use the lengths $p_1, p_2, p_3$ to determine the unknowns $x, y, \theta$.

Finding the position of the platform, given the three strut lengths, is called the forward, or direct, kinematics problem for this manipulator. Namely, the problem is to compute $(x, y)$ and $\theta$ for each given $p_1, p_2, p_3$. Since there are three degrees of freedom, it is natural to expect three numbers to specify the position. For motion planning, it is important to solve this problem as fast as possible, often in real time. Unfortunately, no closed-form solution of the planar Stewart platform forward kinematics problem is known.

The best current methods involve reducing the geometry of Figure 1.14 to a single equation and solving it by using one of the solvers explained in this chapter. Your job is to complete the derivation of this equation and write code to carry out its solution.

Simple trigonometry applied to Figure 1.14 implies the following three equations:

$$
\begin{aligned}
p_1^2 &= x^2 + y^2 \\
p_2^2 &= (x + A_2)^2 + (y + B_2)^2 \\
p_3^2 &= (x + A_3)^2 + (y + B_3)^2.
\end{aligned}
\tag{1.38}
$$

In these equations,

$$
\begin{aligned}
A_2 &= L_3\cos\theta - x_1 \\
B_2 &= L_3\sin\theta \\
A_3 &= L_2\cos(\theta+\gamma) - x_2 = L_2[\cos\theta\cos\gamma - \sin\theta\sin\gamma] - x_2 \\
B_3 &= L_2\sin(\theta+\gamma) - y_2 = L_2[\cos\theta\sin\gamma + \sin\theta\cos\gamma] - y_2.
\end{aligned}
$$

Note that (1.38) solves the inverse kinematics problem of the planar Stewart platform, which is to find $p_1, p_2, p_3$, given $x, y, \theta$. Your goal is to solve the forward problem, namely, to find $x, y, \theta$, given $p_1, p_2, p_3$.

Multiplying out the last two equations of (1.38) and using the first yields

$$p_2^2 = x^2 + y^2 + 2A_2x + 2B_2y + A_2^2 + B_2^2 = p_1^2 + 2A_2x + 2B_2y + A_2^2 + B_2^2$$
$$p_3^2 = x^2 + y^2 + 2A_3x + 2B_3y + A_3^2 + B_3^2 = p_1^2 + 2A_3x + 2B_3y + A_3^2 + B_3^2,$$

which can be solved for $x$ and $y$ as

$$x = \frac{N_1}{D} = \frac{B_3(p_2^2 - p_1^2 - A_2^2 - B_2^2) - B_2(p_3^2 - p_1^2 - A_3^2 - B_3^2)}{2(A_2B_3 - B_2A_3)}$$
$$y = \frac{N_2}{D} = \frac{-A_3(p_2^2 - p_1^2 - A_2^2 - B_2^2) + A_2(p_3^2 - p_1^2 - A_3^2 - B_3^2)}{2(A_2B_3 - B_2A_3)}, \tag{1.39}$$

as long as $D = 2(A_2B_3 - B_2A_3) \neq 0$.

Substituting these expressions for $x$ and $y$ into the first equation of (1.38), and multiplying through by $D^2$, yields one equation, namely,

$$f = N_1^2 + N_2^2 - p_1^2D^2 = 0 \tag{1.40}$$

in the single unknown $\theta$. (Recall that $p_1, p_2, p_3, L_1, L_2, L_3, \gamma, x_1, x_2, y_2$ are known.) If the roots of $f(\theta)$ can be found, the corresponding $x$- and $y$- values follow immediately from (1.39).

Note that $f(\theta)$ is a polynomial in $\sin\theta$ and $\cos\theta$, so, given any root $\theta$, there are other roots $\theta + 2\pi k$ that are equivalent for the platform. For that reason, we can restrict attention to $\theta$ in $[-\pi, \pi]$. It can be shown that $f(\theta)$ has at most six roots in that interval.

### Suggested activities:

1. Write a MATLAB function file for $f(\theta)$. The parameters $L_1, L_2, L_3, \gamma, x_1, x_2, y_2$ are fixed constants, and the strut lengths $p_1, p_2, p_3$ will be known for a given pose. Check Appendix B.5 if you are new to MATLAB function files. Here, for free, are the first and last lines:

```
function out=f(theta)
       :
       :
out=N1^2+N2^2-p1^2*D^2;
```

   To test your code, set the parameters $L_1 = 2$, $L_2 = L_3 = \sqrt{2}$, $\gamma = \pi/2$, $p_1 = p_2 = p_3 = \sqrt{5}$ from Figure 1.15. Then, substituting $\theta = -\pi/4$ or $\theta = \pi/4$, corresponding to Figures 1.15(a, b), respectively, should make $f(\theta) = 0$.

2. Plot $f(\theta)$ on $[-\pi, \pi]$. You may use the @ symbol as described in Appendix B.5 to assign a function handle to your function file in the plotting command. You may also need to precede arithmetic operations with the "." character to vectorize the operations, as explained in Appendix B.2. As a check of your work, there should be roots at $\pm\pi/4$.

3. Reproduce Figure 1.15. The MATLAB commands

```
>> plot([u1 u2 u3 u1],[v1 v2 v3 v1],'r'); hold on
>> plot([0 x1 x2],[0 0 y2],'bo')
```

   will plot a red triangle with vertices `(u1,v1)`, `(u2,v2)`, `(u3,v3)` and place small circles at the strut anchor points `(0,0)`, `(0,x1)`, `(x2,y2)`. In addition, draw the struts.

4. Solve the forward kinematics problem for the planar Stewart platform specified by $x_1 = 5$, $(x_2, y_2) = (0, 6)$, $L_1 = L_3 = 3$, $L_2 = 3\sqrt{2}$, $\gamma = \pi/4$, $p_1 = p_2 = 5$, $p_3 = 3$. Begin

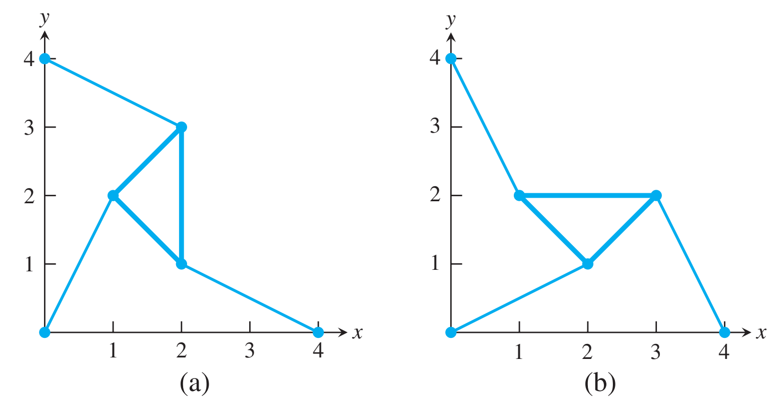


**Figure 1.15 Two poses of the planar Stewart platform with identical arm lengths.** Each pose corresponds to a solution of (1.38) with strut lengths $p_1 = p_2 = p_3 = \sqrt{5}$. The shape of the triangle is defined by $L_1 = 2$, $L_2 = L_3 = \sqrt{2}$, $\gamma = \pi/2$.

by plotting $f(\theta)$. Use an equation solver to find all four poses, and plot them. Check your answers by verifying that $p_1$, $p_2$, $p_3$ are the lengths of the struts in your plot.

5. Change strut length to $p_2 = 7$ and re-solve the problem. For these parameters, there are six poses.

6. Find a strut length $p_2$, with the rest of the parameters as in Step 4, for which there are only two poses.

7. Calculate the intervals in $p_2$, with the rest of the parameters as in Step 4, for which there are $0, 2, 4$, and $6$ poses, respectively.

8. Derive or look up the equations representing the forward kinematics of the three-dimensional, six-degrees-of-freedom Stewart platform. Write a MATLAB program and demonstrate its use to solve the forward kinematics. See Merlet [2000] for a good introduction to prismatic robot arms and platforms. ✔

## Software and Further Reading

There are many algorithms for locating solutions of nonlinear equations. The slow, but always convergent, algorithms like the Bisection Method contrast with routines with faster convergence, but without guarantees of convergence, including Newton's Method and variants. Equation solvers can also be divided into two groups, depending on whether or not derivative information is needed from the equation. The Bisection Method, the Secant Method, and Inverse Quadratic Interpolation are examples of methods that need only a black box providing a function value for a given input, while Newton's Method requires derivatives. Brent's Method is a hybrid that combines the best aspects of slow and fast algorithms and does not require derivative calculations. For this reason, it is heavily used as a general-purpose equation solver and is included in many comprehensive software packages.

MATLAB's `fzero` command implements Brent's Method and needs only an initial interval or one initial guess as input. The ZBREN program of IMSL, the NAG routine c05adc, and `netlib` FORTRAN program fzero.f all rely on this basic approach.