Assignment 8

Reif Problem 5.1

adiabatic expansion of ideal gas:

$$(V_i, T_i) \rightarrow (V_f, T_f)$$

a) We can calculate T_f using

$$pV^{\gamma} = \text{constant}$$

along with the EOS

$$p = \frac{NkT}{V}$$

$$\Rightarrow \quad \frac{NkT}{V} V^{\gamma} = \text{const.}$$

If $N = \text{const.}$, then

$$\boxed{T V^{\gamma - 1} = \text{const.}}$$

Hence we find that

$$T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$$

$$\Rightarrow \quad \boxed{T_f = T_i \left(\frac{V_i}{V_f}\right)^{\gamma-1}}$$

b) next we see if we can get the same result starting with the assumption of constant entropy.

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV = 0$$

$$\Rightarrow \left(\frac{\partial T}{\partial V}\right)_S = -\frac{\left(\frac{\partial S}{\partial V}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_V}$$

next we use the second law to write

$$dQ = T\,dS$$

$$\Rightarrow \left(\frac{\partial S}{\partial T}\right)_V = \frac{1}{T}\left(\frac{\partial Q}{\partial T}\right)_V = \frac{C_V}{T}$$

where C_V is the heat capacity. For an ideal gas,

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \left(\frac{\partial \frac{3}{2}NkT}{\partial T}\right)_V$$

$$\Rightarrow \boxed{C_V = \frac{3}{2}Nk}$$

Next we use one of Maxwell's relations to write

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

For an ideal gas this yields

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial \frac{NkT}{V}}{\partial T}\right)_V = \frac{Nk}{V}$$

Combining results, we find that

$$\left(\frac{\partial T}{\partial V}\right)_S = -\frac{\frac{\cancel{Nk}}{V}}{\frac{1}{T}\frac{3}{2}\cancel{Nk}} = -\frac{2}{3}\frac{T}{V}$$

or

$$\left(\frac{\partial \ln T}{\partial \ln V}\right)_S = -\frac{2}{3}$$

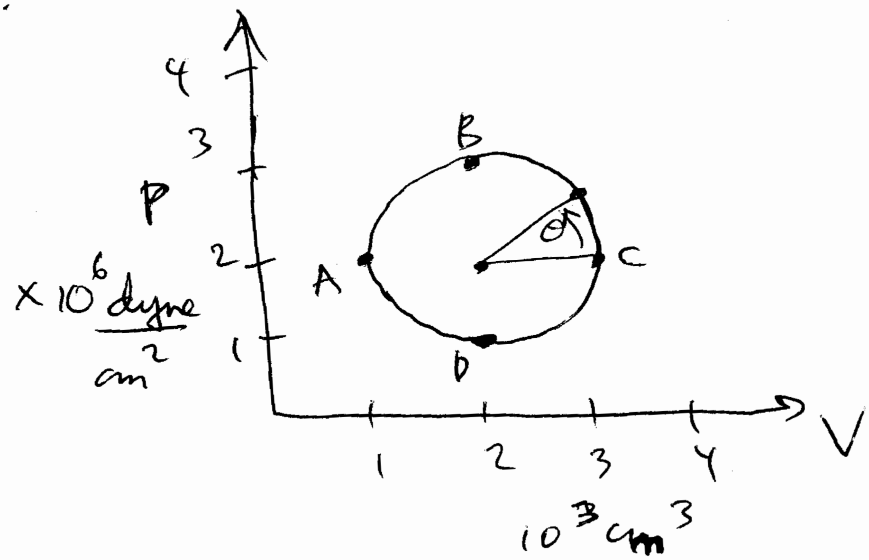
$$\Rightarrow \quad T \propto V^{-2/3}$$

$$\Rightarrow \quad \boxed{T V^{2/3} = \text{constant}}$$

This is consistent with part (a), since $\gamma - 1 = \frac{5}{3} - 1 = \frac{2}{3}$ for an ideal gas.

Problem 5.2

Consider ~~the~~ process indicated in the diagram:



a) work done in one cycle is computed by writing

$$dW = p\,dV$$

$$\therefore \quad \Delta W = \int p\,dV$$

We need to transform to polar coordinates using

$$V = (2 + \cos\theta)\, cm^3\, 10^3$$

$$p = (2 + \sin\theta)\, \frac{dyne}{cm^2}\, 10^6$$

b) Compute the energy difference between state C and state A. Since this is an ideal gas, we have

$$E = \frac{3}{2} NkT \quad , \quad PV = NkT$$

$$\therefore \quad E = \frac{3}{2} PV$$

Therefore we find that

$$E_A = \frac{3}{2} \cdot 2\times10^6 \frac{\text{dynes}}{\text{cm}^2} \cdot 10^3 \text{cm}^3$$

$$= 3\times10^9 \cdot 10^{-5} \text{N} \cdot 10^{-2} \text{m}$$

$$\therefore \quad E_A = 300 \text{ J}$$

Likewise, we find that

$$E_c = \frac{3}{2} \cdot 2\times10^6 \frac{\text{dynes}}{\text{cm}^2} \cdot 3\times10^3 \text{cm}^3$$

$$\therefore \quad E_c = 9\times10^9 \cdot 10^{-5} \text{N} \cdot 10^{-2} \text{m}$$

$$\therefore \quad E_c = 900 \text{ J}$$

The energy difference is therefore

$$\boxed{E_c - E_A = 600 \text{ J}}$$

We obtain

$$\frac{dV}{d\theta} = -\sin\theta \ \mathrm{cm}^3 \, 10^3$$

The integral now becomes for one cycle

$$\Delta W = -\int_{2\pi}^{0} (2+\sin\theta)\, 10^6 \frac{\mathrm{dyne}}{\mathrm{cm}^2} \sin\theta \ 10^3 \mathrm{cm}^3 \, d\theta$$

Note that

$$1 \ \mathrm{dyne} = 1 \, \frac{\mathrm{g\,cm}}{\mathrm{s}^2} = \frac{10^{-3}\,\mathrm{kg}\ 10^{-2}\,\mathrm{m}}{\mathrm{s}^2}$$

$$\therefore \quad 1 \ \mathrm{dyne} = 10^{-5}\,\mathrm{N} \qquad (\mathrm{N} = \text{Newton})$$

We find that

$$\Delta W = 10^9 \ 10^{-5}\,\mathrm{N\,cm} \int_0^{2\pi} (2+\sin\theta)\sin\theta \, d\theta$$

$$= 10^4\,\mathrm{N} \ 10^{-2}\,\mathrm{m} \cdot \pi \qquad (1\ \mathrm{N \cdot m} = 1\,\mathrm{J}$$ ← Joule $)$

$$\therefore \quad \boxed{\Delta W = 314 \ \mathrm{J}}$$

This is the work done by the system in one cycle.

c) The heat absorbed in going from A to C along the path ABC is given by

$$\Delta E = \Delta Q - \Delta W$$

$$\Delta W = -10^4\,N\;10^{-2}\,m \int_{\pi}^{0} (2+\sin\theta)\sin\theta\,d\theta$$

$$= 100\,J \cdot \int_{0}^{\pi} (2+\sin\theta)\sin\theta\,d\theta$$

$$= 100\,J \cdot \left(4 + \frac{\pi}{2}\right)$$

$$\Delta W = 557{,}08\,J$$

Then we have

$$\Delta Q = \Delta E + \Delta W$$

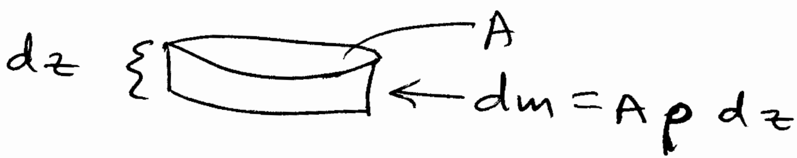
$$= E_c - E_A + 557.08\,J$$

$$= 600\,J + 557.08\,J$$

$$\boxed{\Delta Q = 1157\,J}$$

Reif Problem 5.7

The forces of gravity and pressure must balance:



ρ = mass density,

$$p = \frac{NkT}{V}$$

$$= \frac{\nu R T}{V}\frac{\mu}{\mu}$$

$$= \frac{\rho R T}{\mu} \quad , \quad \rho = \frac{\nu \mu}{V}$$

a) Balance of forces requires that

$$g\, dm = -dp \cdot A$$

Hence

$$dp = -\frac{g A \rho dz}{A}$$

$$= -g\, dz \frac{\mu p}{RT}$$

$$\therefore \quad \boxed{\frac{dp}{p} = -\frac{\mu g}{RT} dz}$$

b) if the pressure change with height is adiabatic, then from problem 5.5 we have

$$T p^{\frac{1-\gamma}{\gamma}} = \text{const.}$$

$$\Rightarrow \quad T^{\frac{\gamma}{1-\gamma}} p = \text{const.}$$

$$\Rightarrow \quad p \propto T^{\frac{\gamma}{\gamma-1}}$$

$$\Rightarrow \quad \boxed{\frac{dp}{p} = \frac{\gamma}{\gamma-1} \frac{dT}{T}}$$

c) Combining (a) and (c) we find that

$$\frac{\gamma}{\gamma-1}\frac{dT}{T} = -\frac{\mu g}{kT}\,dz$$

$$\Rightarrow \quad \boxed{\frac{dT}{dz} = -\left(\frac{\gamma-1}{\gamma}\right)\frac{\mu g}{k}}$$

d) If the atmosphere is <u>isothermal</u>, then (a) yields

$$\int_{p_0}^{p} d\ln p = -\frac{\mu g}{kT}\int_{z_0}^{z} dz \qquad z_0 = \text{height at sea level}$$

$$\Rightarrow \quad \ln(p/p_0) = -\frac{\mu g}{kT}(z-z_0)$$

$$\therefore \quad \boxed{p = p_0\, e^{-\frac{\mu g}{kT}(z-z_0)}}$$

e) If the atmosphere is adiabatic, then

$$\frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}$$

Substitution into the result from (a) yields

$$\frac{dp}{dz} = \frac{-\mu g}{R}\frac{p}{T} = -\frac{\mu g}{R}\frac{p}{T_0}\left(\frac{p}{p_0}\right)^{\frac{1-\gamma}{\gamma}}$$

$$= -\frac{\mu g}{R}\frac{p_0^{\frac{\gamma-1}{\gamma}}}{T_0}\, p^{\frac{1}{\gamma}}$$

$$\Rightarrow \int_{p_0}^{p} p^{-\frac{1}{\gamma}}\, dp = -\frac{\mu g}{R}\frac{p_0^{\frac{\gamma-1}{\gamma}}}{T_0}\int_{z_0}^{z} dz$$

$$\therefore \left.\frac{p^{1-\frac{1}{\gamma}}}{1-\frac{1}{\gamma}}\right|_{p_0}^{p} = -\frac{\mu g}{R}\frac{p_0^{\frac{\gamma-1}{\gamma}}}{T_0}(z-z_0)$$

$$p^{\frac{\gamma-1}{\gamma}} - p_0^{\frac{\gamma-1}{\gamma}} = -\left(\frac{\gamma-1}{\gamma}\right)\frac{\mu g}{R}\frac{p_0^{\frac{\gamma-1}{\gamma}}}{T_0}(z-z_0)$$

a

$$\boxed{\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} - 1 = -\left(\frac{\gamma-1}{\gamma}\right)\frac{\mu g}{RT_0}(z-z_0)}$$