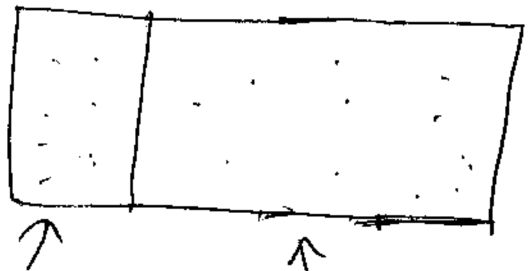
Assignment #5

Reif 3.1

Initial Conditions ⇒

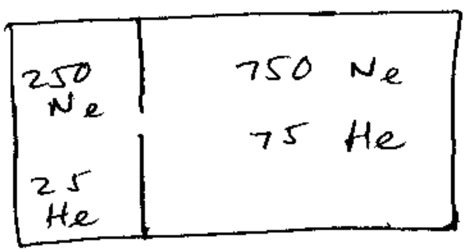


100 molecules of He gas

1000 molecules of Ne gas

A small hole is punctured in the partition. When equilibrium is reached, the density of particles is uniform throughout the box. Hence

(a) The molecules are divided up as follows:



(b) The probability that the initial configuration is re-achieved spontaneously is (assuming the particles don't interact, which is reasonable since the number of molecules is low)

$$P = \left(\frac{3}{4}\right)^{1000}\left(\frac{1}{4}\right)^{100}$$

Using the formula

$$c = \log A^{B}$$

$$10^{c} = A^{B}$$

$$c \ln 10 = B \ln A$$

$$c = B\frac{\ln A}{\ln 10}$$

we find that

$$\log P = \log\left(\frac{3}{4}\right)^{1000} + \log\left(\frac{1}{4}\right)^{100}$$

$$= 1000\,\frac{\ln\left(\frac{3}{4}\right)}{\ln(10)} + 100\,\frac{\ln\left(\frac{1}{4}\right)}{\ln(10)}$$

$$= -125 - 60 = -185$$

$$\boxed{P = 10^{-185}}$$ ← this is $0.\underbrace{00\ldots001}_{184 \text{ zeros!!}}$

<u>very</u> small

Note that P gives the fraction of systems in the ensemble which satisfy the initial conditions after equilibrium is achieved.

Note that this is a very <u>small</u> system, and the results are <u>even more dramatic</u> for macroscopic systems!!

Reif 3.2

(a) $$\beta = \frac{\partial \ln \Omega}{\partial E}$$

$$= \frac{\partial}{\partial E}\left\{N \ln N - \left(\frac{N}{2} - \frac{E}{2\mu H}\right)\ln\left(\frac{N}{2} - \frac{E}{2\mu H}\right) - \left(\frac{N}{2} + \frac{E}{2\mu H}\right)\ln\left(\frac{N}{2} + \frac{E}{2\mu H}\right)\right\}$$

$$= \frac{1}{2\mu H}\ln\left(\frac{N}{2} - \frac{E}{2\mu H}\right) + \cancel{\frac{1}{2\mu H}} - \frac{1}{2\mu H}\ln\left(\frac{N}{2} + \frac{E}{2\mu H}\right) - \cancel{\frac{1}{2\mu H}}$$

$$\therefore \quad \beta = \frac{1}{2\mu H}\left\{\ln\left(\frac{N}{2} - \frac{E}{2\mu H}\right) - \ln\left(\frac{N}{2} + \frac{E}{2\mu H}\right)\right\}$$

$$= \frac{1}{2\mu H}\ln\left(\frac{\frac{N}{2} - \frac{E}{2\mu H}}{\frac{N}{2} + \frac{E}{2\mu H}}\right)$$

or

$$\boxed{\beta = \frac{1}{2\mu H}\ln\left(\frac{N\mu H - E}{N\mu H + E}\right)}$$

Relating this to the temperature yields

$$\frac{1}{kT} = \frac{1}{2\mu H}\ln\left(\frac{N\mu H - E}{N\mu H + E}\right)$$

$$\therefore \quad e^{\frac{2\mu H}{kT}} = \frac{N\mu H - E}{N\mu H + E}$$

$\therefore$

$$E = \frac{N\mu H - N\mu H e^{\frac{2\mu H}{kT}}}{1 + e^{\frac{2\mu H}{kT}}}$$

$$= N\mu H \left(\frac{e^{-\frac{\mu H}{kT}} - e^{\frac{\mu H}{kT}}}{e^{-\frac{\mu H}{kT}} + e^{\frac{\mu H}{kT}}} \right)$$

$$= -N\mu H \left(\frac{e^{\frac{\mu H}{kT}} - e^{-\frac{\mu H}{kT}}}{e^{\frac{\mu H}{kT}} + e^{-\frac{\mu H}{kT}}} \right)$$

$\therefore$

$$\boxed{E = -N\mu H \tanh\left(\frac{\mu H}{kT}\right)}$$

(b) If $E > 0$ then $T < 0$

(c)

$$E = -(n_{up} - n_{down})\mu H$$

$$M = (n_{up} - n_{down})\mu$$

$\therefore$ $$\boxed{M = -\frac{E}{H} = N\mu \tanh\left(\frac{\mu H}{kT}\right)}$$

Reif 3.3

Consider two systems A and A' with N and N' particles with magnetic moments μ and μ' respectively. They are placed in thermal contact with each other in an external magnetic field H. Initially, the energies of systems A and A' are $bN\mu H$ and $b'N'\mu'H$ where $|b| \ll 1$ and $|b'| \ll 1$ so that the results of problem 2.4c can be applied.

(a) First we calculate β using

$$\beta \equiv \frac{d \ln \Omega}{dE}$$

using the results of problem 2.4c we have

$$\beta = \frac{d}{dE} \ln e^{\frac{-E^2}{2N(\mu H)^2}}$$

$$= \frac{d}{dE} \; \frac{-E^2}{2N(\mu H)^2} = -\frac{E}{N(\mu H)^2}$$

$$\Rightarrow \boxed{\beta = \frac{-E}{N(\mu H)^2} = \frac{1}{kT}}$$

for system A and

$$\boxed{\beta' = \frac{-E'}{N'(\mu' H)^2} = \frac{1}{kT'}}$$

for system A'. When the systems are in equilibrium, the two temperatures are equal and $E = \tilde{E}$, $E' = \tilde{E}'$, where $\tilde{E}$ and $\tilde{E}'$ are the most probable energies of systems A and A', respectively. Hence we obtain

$$\frac{-\tilde{E}}{N(\mu H)^2} = \frac{-\tilde{E}'}{N'(\mu' H)^2}$$

$$\Rightarrow \boxed{\frac{\tilde{E}}{N\mu^2} = \frac{\tilde{E}'}{N'\mu'^2}}$$

(b) Next we solve for the energy $\tilde{E}$ using the energy conservation principle, which implies that

$$\tilde{E} + \tilde{E}' = E^{(0)} = bN\mu H + b'N'\mu' H$$

Substitution yields

$$\tilde{E} + \frac{N'\mu'^2}{N\mu^2}\tilde{E} = bN\mu H + b'N'\mu' H$$

$$\therefore \quad \tilde{E} = \frac{bN\mu H + b'N'\mu' H}{1 + \dfrac{N'\mu'^2}{N\mu^2}}$$

$$\therefore \quad \boxed{\tilde{E} = N\mu^2\left(\frac{bN\mu H + b'N'\mu' H}{N\mu^2 + N'\mu'^2}\right)}$$

(c) The heat Q absorbed by A in going from the initial situation to equilibrium is given by

$$Q = \tilde{E} - bN\mu H$$

$$= \frac{\cancel{N\mu^2 bN\mu H} + N\mu^2 b'N'\mu' H - \cancel{bN\mu H N\mu^2} - bN\mu H N'\mu'^2}{N\mu^2 + N'\mu'^2}$$

$$\therefore \quad \boxed{Q = NN'H\left(\frac{b'\mu^2\mu' - b\mu\mu'^2}{N\mu^2 + N'\mu'^2}\right)}$$

(d) The probability distribution for E is given by the distribution of the number of states accessible by the combined system. We have

$$\Omega^{(0)} = \Omega(E)\,\Omega'(E')$$
$$\propto \quad e^{\frac{-E^2}{2N(\mu H)^2}}\; e^{\frac{-E'^2}{2N'(\mu' H)^2}}$$

where
$$E + E' = E^{(0)} = bN\mu H + b'N'\mu' H$$

by energy conservation. The probability that the energy of A falls between E and $E+dE$ is given by

$$P(E)\,dE = B \exp\left[\frac{-E^2}{2N(\mu H)^2} - \frac{(bN\mu H + b'N'\mu' H - E)^2}{2N'(\mu' H)^2}\right] dE$$

where B is a constant which is determined by the normalization requirement
$$\int_0^{E^{(0)}} P(E)\,dE = 1$$

Using our result for $\tilde{E}$ we can write

$$P(E)\,dE = B \exp\left\{\frac{-E^2}{2N\mu^2 H^2} - \frac{\left[\tilde{E}\left(1+\frac{N'\mu'^2}{N\mu^2}\right) - E\right]^2}{2N'\mu'^2 H^2}\right\} dE$$

$$= B \exp\left\{E^2\left[-\frac{1}{2N\mu^2H^2} - \frac{1}{2N'\mu'^2H^2}\right] + E\left[\frac{2\tilde{E}\left(1+\frac{N'\mu'^2}{N\mu^2}\right)}{2N'\mu'^2H^2}\right] + \text{const.}\right\} dE$$

this depends on $\tilde{E}$ but not

$\therefore$

$$P(E)dE = B_o \exp\left\{E^2\left[\frac{-N'\mu'^2 - N\mu^2}{2NN'\mu^2\mu'^2H^2}\right] + E\left[\frac{\tilde{E}\,(N\mu^2+N'\mu'^2)}{N\mu^2N'\mu'^2H^2}\right]\right\}dE$$

where B_o is another constant. We can rewrite this as

$$P(E)dE = B_o \exp\left\{-\left(\frac{N\mu^2+N'\mu'^2}{2NN'\mu^2\mu'^2H^2}\right)\left[E^2 - E\,\tilde{E}\left(\frac{\cancel{N\mu^2+N'\mu'^2}}{\cancel{NN'\mu^2\mu'^2H^2}}\right)\left(\frac{\cancel{2NN'\mu^2\mu'^2H^2}}{\cancel{N\mu^2+N'\mu'^2}}\right)\right]\right\}dE$$

$$= B_o \exp\left\{-\left(\frac{N\mu^2+N'\mu'^2}{2NN'\mu^2\mu'^2H^2}\right)(E^2 - 2\tilde{E}E)\right\}dE$$

$$P(E)dE = B_o \exp\left\{-\left(\frac{N\mu^2+N'\mu'^2}{2NN'\mu^2\mu'^2H^2}\right)(E^2 - 2\tilde{E}E + \tilde{E}^2 - \tilde{E}^2)\right\}dE$$

where we have completed the square by adding and subtracting $\tilde{E}^2$. This yields the simpler form

$$P(E)dE = B_o \exp\left\{-\left(\frac{N\mu^2+N'\mu'^2}{2NN'\mu^2\mu'^2H^2}\right)(E-\tilde{E})^2 + \text{const.}\right\}dE$$

this depends on $\tilde{E}$ but not on E

Hence we obtain

$$\boxed{P(E)dE = \frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{(E-\tilde{E})^2}{2\sigma^2}}\,dE}$$

where

$$\boxed{\sigma^2 \equiv \frac{NN'\mu\mu'^2H^2}{N\mu^2+N'\mu'^2}}$$

and the constant factor has been set to satisfy the normalization requirement.

(e) The dispersion of the ~~eq~~ energy A is obtained from the probability distribution P(E), and is equal to

$$\boxed{\sigma^2}$$

(f) The relative energy spread is given by

$$\frac{\sigma}{\tilde{E}} = \mu\mu' H \sqrt{\frac{NN'}{N\mu^2 + N'\mu'^2}} \; \frac{N\mu^2 + N'\mu'^2}{bN\mu H + b'N'\mu' H} \; \frac{1}{N\mu^2}$$

As $\frac{N'}{N} \to \infty$ we obtain

$$\frac{\sigma}{\tilde{E}} \to \frac{\mu' \cancel{H}}{N\mu} \sqrt{\frac{N\cancel{N'}}{\cancel{\mu'^2}\cancel{N'}}} \; \frac{\cancel{\mu'^2}\cancel{N'}}{b'\cancel{\mu'}\cancel{H}\cancel{N'}}$$

$$\boxed{\frac{\sigma}{\tilde{E}} \to \frac{\mu'}{\mu} \frac{1}{b'} \frac{1}{\sqrt{N}}}$$

Reif 3.4

System A is placed in contact with a heat reservoir A' at temperature T' and absorbs heat Q. Then ~~the~~ entropy change of A is given by

$$\Delta S = \int dS = \int \frac{dQ}{T}$$

Assuming that A is initially cooler than A', then Q > 0 and

$$T \leq T'$$

with equality occurring at equilibrium, after heat Q has been transferred from A' to A. Hence we have

$$\Delta S = \int \frac{dQ}{T} > \int \frac{dQ}{T'} = \frac{Q}{T'}$$

where the last step follows since T' is a constant for a heat reservoir. Hence we have proven that

$$\boxed{\Delta S \geq \frac{Q}{T'}}$$

Note that equality occurs if T is essentially equal to T'.