Assignment #4

Reif 2.1

Particle with position x and momentum p.

Position lies between $x=0$ and $x=L$, energy lies between E and $E+\delta E$.

Then since

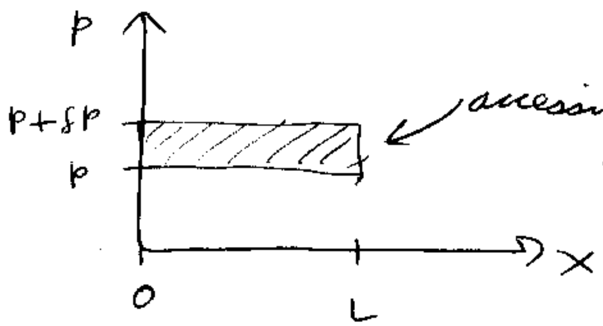
$$E = \frac{1}{2} m v^2 = \frac{1}{2} m \left(\frac{p}{m}\right)^2 = \frac{p^2}{2m}$$

we have

$$p = \sqrt{2mE}$$

$$\delta p = \sqrt{2m}\ \frac{1}{2} E^{-1/2} \delta E$$

with the phase space diagram



Reif 2.4

N weakly interacting particles with n_1 aligned with $\vec{H}$ and n_2 antiparallel to $\vec{H}$. Then the energy of the system is given by

$$E = -(n_1 - n_2)\mu H \quad , \quad n_1 + n_2 = N$$

$$\Rightarrow \quad E = -(n_1 + n_1 - N)\mu H = (N - 2n_1)\mu H$$

$$n_1 = \frac{N - \frac{E}{\mu H}}{2} = \frac{N}{2} - \frac{E}{2\mu H}$$

a) Assuming that the energy is between E and $E+\delta E$ (where $\delta E \ll E$), find the total number of states $\Omega(E)$ in this energy range. We have

$$\Delta E = 2\mu H$$

Therefore

$$\frac{\delta E}{2\mu H}$$

is the number of energy <u>levels</u>. Each level has associated with it a number of states equal to

$$\frac{N!}{n_1!\,(N-n_1)!} = \frac{N!}{\left(\frac{N}{2}-\frac{E}{2\mu H}\right)!\,\left(\frac{N}{2}+\frac{E}{2\mu H}\right)!}$$

Therefore the total number of states in the energy range δE is equal to

$$\boxed{\Omega(E) = \frac{N!}{\left(\frac{N}{2}-\frac{E}{2\mu H}\right)!\,\left(\frac{N}{2}+\frac{E}{2\mu H}\right)!}\,\frac{\delta E}{2\mu H}}$$

b) $\ln\Omega(E) \simeq \ln N! - \ln\left(\frac{N}{2}-\frac{E}{2\mu H}\right)! - \ln\left(\frac{N}{2}+\frac{E}{2\mu H}\right)!$

$$\simeq N\ln N - \cancel{N} - \left(\frac{N}{2}-\frac{E}{2\mu H}\right)\ln\left(\frac{N}{2}-\frac{E}{2\mu H}\right)$$
$$+ \cancel{\frac{N}{2}} - \cancel{\frac{E}{2\mu H}} - \left(\frac{N}{2}+\frac{E}{2\mu H}\right)\ln\left(\frac{N}{2}+\frac{E}{2\mu H}\right)$$
$$+ \cancel{\frac{N}{2}} + \cancel{\frac{E}{2\mu H}}$$

$\therefore$

$$\boxed{\ln\Omega(E) \simeq N\ln N - \left(\frac{N}{2}-\frac{E}{2\mu H}\right)\ln\left(\frac{N}{2}-\frac{E}{2\mu H}\right) - \left(\frac{N}{2}+\frac{E}{2\mu H}\right)\ln\left(\frac{N}{2}+\frac{E}{2\mu H}\right)}$$

c) Make Gaussian approximation for $\Omega(E)$, assuming that $|E| \ll N\mu H$.

Differentiation of our result for $\ln \Omega$ yields

$$\frac{d \ln \Omega}{dE} = \frac{1}{2\mu H} \ln\left(\frac{N}{2} - \frac{E}{2\mu H}\right) + \cancel{\frac{1}{2\mu H}}$$
$$- \frac{1}{2\mu H} \ln\left(\frac{N}{2} + \frac{E}{2\mu H}\right) - \cancel{\frac{1}{2\mu H}}$$

$$\boxed{\frac{d \ln \Omega}{dE} = \frac{1}{2\mu H}\left[\ln\left(\frac{N}{2} - \frac{E}{2\mu H}\right) - \ln\left(\frac{N}{2} + \frac{E}{2\mu H}\right)\right]}$$

This vanishes when $E = 0$. Hence we need to evaluate the second derivative of $\ln \Omega$ with respect to E at $E = 0$. We have

$$\left.\frac{d^2 \ln \Omega}{dE^2}\right|_{E=0} = \frac{1}{2\mu H}\left.\left[\frac{-\frac{1}{2\mu H}}{\frac{N}{2} - \frac{E}{2\mu H}} - \frac{\frac{1}{2\mu H}}{\frac{N}{2} + \frac{E}{2\mu H}}\right]\right|_{E=0}$$

$$= \frac{1}{\cancel{2}\mu H}\left(-\frac{\cancel{2}}{N} \frac{1}{\cancel{2}\mu H} \cdot \cancel{2}\right) = -\frac{1}{N(\mu H)^2}$$

Hence our second-order Taylor expansion for $\ln \Omega$ looks like

$$\ln \Omega = \text{const.} + 0\, E - \frac{1}{N(\mu H)^2} \frac{E^2}{2}$$

$$\Rightarrow \boxed{\Omega(E) \propto e^{-\frac{E^2}{2N(\mu H)^2}}}$$