Reif 1.9

The binomial prob. distribution is given by

$$P(n,N) = p^n (1-p)^{N-n} \frac{N!}{n!(N-n)!}$$

We are interested in this distribution in the limit of small p and $n \ll N \gg 1$.

a) we have $\ln(1-p) \simeq -p$ for $p \ll 1$.

Then

$$(1-p)^{N-n} = e^{\ln(1-p)^{N-n}} = e^{(N-n)\ln(1-p)}$$

$$\simeq e^{(N-n)(-p)} \simeq e^{-Np} \quad , \; \begin{matrix} N \gg n \\ p \ll 1 \end{matrix}$$

b) $$\frac{N!}{(N-n)!} = \frac{N(N-1)(N-2)\cdots 2\cdot 1}{(N-n)(N-n-1)\cdots 2\cdot 1}$$

$$= \underbrace{N(N-1)(N-2)\cdots(N-n+1)}_{n \text{ factors}} \quad , N \gg n$$

$$\simeq N^n$$

c) Hence

$$p(n,N) \simeq p^n e^{-Np} \frac{N^n}{n!}$$

or $$\boxed{p(n,N) \simeq \frac{\lambda^n e^{-\lambda}}{n!}} \quad , \; \lambda \equiv pN = \bar{n}$$

which is the Poisson distribution.

Assignment # 3

Reif 1.10

Consider the Poisson distribution

$$P(n) = \frac{\lambda^n e^{-\lambda}}{n!}$$

a) Show that $\sum_{n=0}^{N} P(n) = 1$

Extending the sum to infinity yields

$$\sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{-\lambda} \cdot e^{\lambda} = 1 \quad \checkmark$$

b) Calculate $\bar{n}$ using the Poisson distribution.

We have

$$\bar{n} = \sum_{n=0}^{N} P(n)\, n = \sum_{n=0}^{N} \frac{\lambda^n e^{-\lambda}}{n!} \cdot n$$

$$= \sum_{n=1}^{N} \frac{\lambda^n e^{-\lambda}}{(n-1)!} = \lambda \sum_{n=1}^{N} \frac{\lambda^{n-1} e^{-\lambda}}{(n-1)!}$$

$$= \lambda e^{-\lambda} \sum_{n=1}^{N} \frac{\lambda^{n-1}}{(n-1)!} \approx \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!}$$

$$\approx \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \approx \lambda e^{-\lambda} \cdot e^{\lambda}$$

$$\bar{n} \approx \lambda = pN$$

$$\Rightarrow \boxed{\bar{n} = pN} \quad \checkmark$$

or

$$\boxed{\bar{n} = \lambda}$$

c) Calculate $\overline{\Delta n^2} \equiv \overline{(n-\bar{n})^2} = \sigma^2$

First note that

$$\sigma^2 = \overline{\Delta n^2} = \overline{(n-\bar{n})^2} = \overline{n^2 - 2n\bar{n} + \bar{n}^2}$$

$$= \overline{n^2} - 2\bar{n}\,\bar{n} + \bar{n}^2$$

$$\therefore \quad \overline{\Delta n^2} = \overline{n^2} - \bar{n}^2$$

Therefore we need to calculate $\overline{n^2}$ using

$$\overline{n^2} = \sum_{n=0}^{N} n^2 \frac{\lambda^n e^{-\lambda}}{n!} \approx e^{-\lambda} \sum_{n=1}^{\infty} \frac{n\lambda^n}{(n-1)!}$$

$$\approx e^{-\lambda} \lambda \sum_{n=1}^{\infty} \frac{n\lambda^{n-1}}{(n-1)!} \approx e^{-\lambda} \lambda \sum_{n=1}^{\infty} \frac{d}{d\lambda} \frac{\lambda^n}{(n-1)!}$$

$$\approx e^{-\lambda} \lambda \frac{d}{d\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{m!}$$

$$\approx e^{-\lambda} \lambda \frac{d}{d\lambda} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \approx e^{-\lambda} \lambda \frac{d}{d\lambda} (\lambda e^{\lambda})$$

$$\approx \cancel{e^{-\lambda}} \lambda (\lambda \cancel{e^{\lambda}} + 1\cancel{e^{\lambda}}) \approx \lambda(\lambda + 1)$$

Hence we obtain

$$\overline{\Delta n^2} = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

or $\quad \boxed{\overline{\Delta n^2} = \lambda} = \sigma^2$

Alternative: Compute ~~the~~ variance of ~~the~~ Poisson distribution using

$$\overline{n^2} = \sum_{n=0}^{N} n^2 \frac{\lambda^n e^{-\lambda}}{n!} = \sum_{n=1}^{N} \frac{\lambda^n e^{-\lambda} n}{(n-1)!}$$

$$= e^{-\lambda} \sum_{n=1}^{N} \left[\frac{\lambda^n}{(n-1)!}(n-1) + \frac{\lambda^n}{(n-1)!} \right]$$

$$= e^{-\lambda} \sum_{n=2}^{N} \frac{\lambda^n}{(n-2)!} + e^{-\lambda} \sum_{n=1}^{N} \frac{\lambda^n}{(n-1)!}$$

$$= e^{-\lambda} \lambda^2 \sum_{n=2}^{N} \frac{\lambda^{n-2}}{(n-2)!} + e^{-\lambda} \lambda \sum_{n=1}^{N} \frac{\lambda^{n-1}}{(n-1)!}$$

$$= e^{-\lambda} \lambda^2 \sum_{m=0}^{N-2} \frac{\lambda^m}{m!} + e^{-\lambda} \lambda \sum_{m=0}^{N-1} \frac{\lambda^m}{m!}$$

$$\approx e^{-\lambda} \lambda^2 \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} + e^{-\lambda} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!}$$

$$\overline{n^2} \approx \cancel{e^{-\lambda}} \lambda^2 \cancel{e^{\lambda}} + \cancel{e^{-\lambda}} \lambda \cancel{e^{\lambda}}$$

Hence we find ~~that~~

$$\overline{n^2} \approx \lambda^2 + \lambda$$

Recall ~~that~~ $\sigma^2 = \overline{n^2} - \overline{n}^2$

$$\therefore \sigma^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\Rightarrow \boxed{\sigma^2 = \lambda}$$

Reif 1.14

A penny is tossed 400 times.
The probability of getting 215 heads is given by the Gaussian approximation,

$$P(n,N) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(n-\bar{n})^2}{2\sigma^2}}$$

where $\bar{n} = pN$, $\sigma^2 = Npq$, $q = 1-p$, $p = \frac{1}{2}$, $N = 400$

We have $\sigma^2 = 100$, $\bar{n} = 200$,

$$P(215, 400) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{(215-200)^2}{2\cdot 100}}$$

$$\boxed{P(215,400) \approx 1.3 \times 10^{-2}}$$

Reif 1.16

a) $\bar{N} = \frac{N_0}{V_0} \cdot V$

b) $p = \frac{V}{V_0}$, $\sigma^2 = N_0 p q$

$q = 1 - \frac{V}{V_0}$, $\frac{\sigma^2}{\bar{N}^2} = \frac{N_0 pq}{\left(\frac{N_0 V}{V_0}\right)^2} = \frac{\cancel{N_0} \frac{\cancel{V}}{\cancel{V_0}}\left(1-\frac{V}{V_0}\right)}{\frac{\cancel{N_0 V}}{\cancel{V_0}}} \frac{V_0}{N_0 V}$

$$\therefore \boxed{\frac{\sigma^2}{\bar{N}^2} = \left(1 - \frac{V}{V_0}\right)\frac{1}{\bar{N}}}$$

c) when $V \ll V_0$, part (b) becomes $\frac{\sigma^2}{\bar{N}^2} = \frac{1}{\bar{N}}$ (consistent with Poisson dist with $p \ll 1$)

d) when $V \to V_0$, the dispersion should vanish. This is consistent with the result from part (b).