On White Dwarfs

Oscar Olmedo

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On the beginning of time, a theory states that the universe came from a single point called a singularity, where all matter began to expand. This matter according to the theory began to, in inhomogeneous areas of higher mass density, collapse into itself forming what we now see as galaxies, stars and planets. A topic of much interest in astronomy is the evolution of stars, in particular the dieing stages where a star has several paths as to what it can become due to its mass, after brilliantly shining, like the sun currently is, for millions and billions of years. One of the later evolved stages for a star is called a white dwarf. It is determined by a limit which has been set on the mass of stars, where if not exceeded, this may be the fate for the star. It is thought that this will one day be the fate of our own sun.

The births of white dwarfs begin at the core of stars comparable to the mass of the sun. Upon burning hydrogen, and when the density and temperature at the core of a star reach a certain level, helium fusion begins and becomes the dominant form of energy in the core. At these high pressures a limit is reached called *electron degeneracy* where the electrons at the core can no longer be compressed further. This electron degenerate pressure is what holds the star together, and keeps it from collapsing under its own gravity. Eventually, due to the thermal energy being given of by the core, the outer layers of the star begin to expand. This envelope of the outer layers is eventually ejected as a planetary nebulae because of the violence occurring at the core caused by the high pressure and density.

Calculations to model a white dwarf begin with the electron degeneracy pressure, which can be calculated from the Fermi energy

$$E_f = \hbar^2/2m_e\,(3\pi^2 n_e)^{2/3} \qquad (1)$$

where m_e is the electron mass and n_e is the electron density. This gives the Fermi momentum;

$$p_f = \sqrt{(2m_eE_f)} = h(3n_e/8\pi)^{1/3} \qquad (2)$$

Within white dwarfs of lower mass an approximation assuming that electrons are non-relativistic can be used, but for high mass white dwarfs relativistic effects must be taken into consideration. A white dwarf could be thought of as a gas cloud of electrons where the internal energy is given by

$$E = \int \varepsilon_p f(\varepsilon_p) g(p)dp \qquad (3)$$

ε_p is the energy of a particle of mass m and momentum p and it is assumed that the Fermi momentum $p_f << mc$

$$\varepsilon_p = p^2/2m + mc^2 \qquad (4)$$

$f(\varepsilon_p)$ is the average number of particles in a given state with energy ε_p which follows from Fermi-Dirac statistics where identical Fermions (electrons) must comply with the Pauli exclusion principle which states that only one particle can be in a given quantum state

$$f(\varepsilon_p) = 1 / (\exp[(\varepsilon_p - e_c)/kT]+1) \qquad (5)$$

e_c is the chemical potential energy of the system, k the boltzman constant and T the temperature. Finally g(p)dp is the number of quantum states with momentum between p and p+dp

$$g(p)dp = (g_sV/h^3)\,4\pi p^2\,dp \qquad (6)$$

g_s is the number of independet polarizations of the particle, for Fermions spin ½ particles $g_s = 2$. Solving equation (6) for the integration limits of 0 and p_f

$$N = (8\pi V p_f^3) / (3h^3)$$

At the limit of degenerate electron pressure e_c is equal to the Fermi energy

$$f(\varepsilon_p) = 1 \text{ if } \varepsilon_p \le E_f \quad \text{and} \quad f(\varepsilon_p) = 0 \text{ if } \varepsilon_p > E_f$$

Equation (3), with the defined quantities (4) – (6) becomes

$$E = \int \varepsilon_p(g_sV/h^3)\, 4\pi p^2\, dp \text{ integrated between 0 and } p_f$$

$$E = (8\pi V)/h^3\, [p_f^5/10m + p_f^3mc^2/3]$$

$$\text{Reduces to } E = N[mc^2 + 3p_f^2/10m] \qquad (7)$$

For non-relativistic particles it is found that the pressure P = 2/3 of the kinetic energy density[1]. By taking the kinetic energy part of equation (7) and applying this rule

$$P_{n-r} = n_ep_f^2 / 5m_e \qquad (8)$$

Substituting equation (2) into this the non-relativistic degenerate gas pressure is found

$$P_{n-r} = (n_e^{5/3}\, h^2(3/8\pi)^{2/3})/ 5m_e \qquad (9)$$

Solving for higher mass stars where relativistic properties are a concern and assuming that the Fermi momentum p_f approaches mc, ε_p becomes

$$\varepsilon_p = pc \qquad (10)$$

and solving equation (3) with equation (10) instead of (4) and integrating between 0 and p_f one finds the internal energy to be

$$E = 2Vp_fc/h^3$$

$$E = Np_fc3/4 \qquad (11)$$

with corresponding pressure

$$P_r = n_ep_fc/4 \qquad (12)$$

Substituting equation (2) into this the expression for the relativistic degenerate gas pressure is found

$$P_r = n_e^{4/3}h(3/8\pi)^{1/3}(c/4) \qquad (13)$$

Through the use of a polytrope model[2] which relates pressure P, density ρ and kinetic energy K in the following way

$$P = K\rho^{(n+1)/n} \qquad (14)$$

,where n is the index number that refines the model, and the differential

$$1/r^2 d/dr[r^2/\rho\, d/dr(K\rho^{(n+1)/n})] = -4\pi G\rho \qquad (15)$$

an accurate model of hydrostatic equilibrium within a star can be made.

A model based on the polytrope model proposed by D. D. Clayton is of particular significance. One begins with a solution of the form P=P(r) and on such a solution can impose the limits of hydrostatic equilibrium. Beginning with the fact that the pressure at the center of a star is much larger than the average pressure of the entire system, and also that the pressure gradient dP/dt is zero at the center. With these limits the pressure gradient in a star may follow

$$dP/dr = -(4\pi/3)G\rho_c^2 r\exp(-r^2/a^2) \qquad (16)$$

and the pressure P(r) follows after integration from R to r where r is the distance from the center

$$P(r) = (2\pi/3)G\rho_c^2a^2[\exp(-r^2/a^2)- \exp(-R^2/a^2)] \qquad (17)$$

Here ρ_c is the central density, R the maximum radius, and a the length parameter which is small compared to R. For a star with a large central density, a is approximated by the polytrope model

$$a \approx (3M / (4\pi\rho_c\sqrt{6}))^{1/3} \qquad (18)$$

An approximation for the central pressure can be made with the use of (18), setting r = 0 and neglecting $-\exp(R^2/a^2)$

$$P_c \approx [\pi/36]^{1/3}GM^{2/3}\rho_c^{4/3} \approx 0.44GM^{2/3}\rho_c^{4/3} \qquad (19)$$

A term Y_e is introduced which is the number of electrons per nucleon and is approximately $[1+X_1]/2$ where X_1 is the mass fraction of hydrogen in the volume, for our case of a white dwarf $Y_e = 0.5$. The electron density n_e can then be rewritten as

$$n_e = Y_e\rho_c/m_H \qquad (20)$$

where m_H is the mass of the hydrogen atom i.e. the proton.

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[1] A.C. Phillips 1994 pg.48-49

[2] The polytrope model is beyond the scope of this paper, see A.C. Phillips (1994) for a complete derivation, and approximations, of the polytrope model sec. 5.2

By setting (13) the degenerate relativistic electron pressure and (19) the hydrostatic equilibrium pressure equal to each other and substituting (20) into (13) one finds that the central density ρ_c cancels and one can then solve for the limiting mass of the system[3]. This limiting mass was first studies by S. Chandrasekhar and named after him

$$M_{CH} = 1.072\ M_0 \tag{21}$$

and M_0 is the solar mass

This limit of $M_{CH} = 1.072\ M_0$ found only holds true for white dwarfs entirely composed of degenerate relativistic electron pressure. A more accurate model then needs to be derived that incorporates both relativistic and non-relativistic effects. We begin with the relation

$$P = -\partial E/\partial V \tag{22}$$

and substitute equation (3) for E into this

$$P = \int (d\varepsilon_p/dV) f(\varepsilon_p) g(p) dp \tag{23}$$

It follows that $d\varepsilon_p/dV$ must then be solved for

$$d\varepsilon_p/dV = (d\varepsilon_p/dp)(dp/dV) \tag{24}$$

in this case the energy is determined by special relativity for particles of mass m, momentum p and velocity v, where the relation for (27) comes from the classical view of kinetic energy given by $K_c = p^2/2m$ and dK/dp = velocity v and the ratio between $\varepsilon = \gamma mc^2$ and $p = \gamma mv$.

From $\varepsilon = \gamma mc^2$ we get (26)

$$\varepsilon_p^{\ 2} = p^2c^2 + m^2c^4 \tag{26}$$

$$d\varepsilon_p/dp = v = pc^2/\varepsilon_p \tag{27}$$

From the de Broglie relation $p = h/\lambda = \hbar k$, λ is the wavelength and k the wave number which from quantum theory for a particle in a box $k \propto 1/L$, where L is the length of the potential, and the volume $V \propto L^3$, from these relations comes

$$dp/dV = -p/3V \tag{28}$$

Combining (5), (6), recalling that $g_s = 2$, and (24) through (28), (23) becomes

$$P = (8\pi/3h^3) \int p^4c^2/(m_e^{\ 2}c^4+p^2c^2)^{1/2}\ dp \tag{29}$$

integrated between 0 and p_f

Setting[4] $x = p/m_ec$ (29) reduces to

$$P = (8\pi\ m_e^{\ 4}c^5/3h^3) \int x^4/(1+x^2)^{1/2}\ dx \tag{30}$$

integrated between 0 and $x_f = p_f/m_ec$

And with (2) and (20)

$$x_f = [3Y_e\rho_c/8\pi\ m_H]^{1/3}\ (h/\ m_ec) \tag{31}$$

Solving[5] for P

$$P = (8\pi\ m_e^{\ 4}c^5/3h^3)\ X(x) \tag{32}$$

$$X(x) = (3/8)[x(2/3x^2-1)(1+x^2)^{1/2} + \ln(x+(1+x^2)^{1/2})] \tag{33}$$

Putting P in terms of the relativistic equation (13)

$$P = P_rY(x) \tag{34}$$

$$Y(x) = (4/x^4)X(x) \tag{35}$$

Equation (34) shows a more accurate model of the electron degenerate pressure inside a white dwarf in comparison to (13) and (9) which only take into account either only relativistic or non-relativistic effects.

Returning to the polytrope model of hydrostatic equilibrium, once solved for with the conditions set in (14) through (17), a more accurate calculation than (19) is made. This solution is found with an index of n = 3

$$P_c = 0.36\ GM^{2/3}\rho_c^{\ 4/3} \tag{36}$$

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[3] See Appendix 2. for the derivation

[4] This x term is introduced by A.C. Phillips (1994) to simplify the integral

[5] See Appendix 3. for the integral identities used to solve (30)

It can be seen that the only difference between this more accurate result (36) and the approximation (19) only differ by a ratio of 0.36/0.44 = 0.81 or a percent difference of about 22%.

To find the limiting mass for this new model, equations (34) and (36) are set equal to each other as was done to find (21). This now yields

$M = 1.46\ M_0\ [Y(x)]^{3/2}$ (37)

(21) becomes[6] $M_{CH} = 1.46\ M_0$ (38)

Figure 1. The Chandrasekhar mass limit

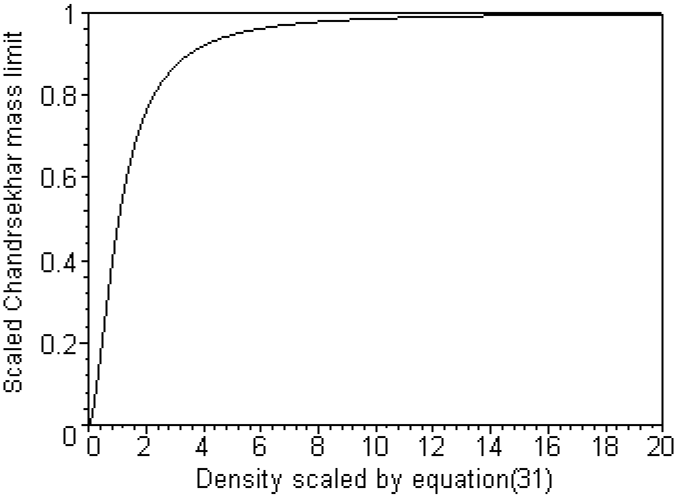

Figure 1. shows the plot of equation (37) where the mass is in units of M_{CH} and the density is scaled with respect to (31). It can be seen that as the density increases the mass reaches the limit of which this star can not exceed because the density must go to infinity to reach it. As stated before this limit is called the Chandrasekhar mass limit.

The next important aspect of the White Dwarf is its maximum Radius for a given mass. This radius can be calculated with the fact that in equation (9) n_e is raised to the 5/3 power setting this equation equal to (36) the central density can be solved[7] for as it does not cancel due to that fact that in (36) ρ_c is raised to the 4/3 power. With this the average density $<\rho>$ can be calculated[8] and is equal to $\rho_c/6$. With the following identities; $<\rho>V = M$ and $V = 4/3\pi R^3$,an approximation for the radius can be made

$R = [3M/(4\pi<\rho>)]^{1/3}$ (39)

With the results of Appendix 4.

$R = (18/(4\pi 8.17_{10}^{-52}M))^{1/3}$ (40)

Figure 2. Radius of White dwarf

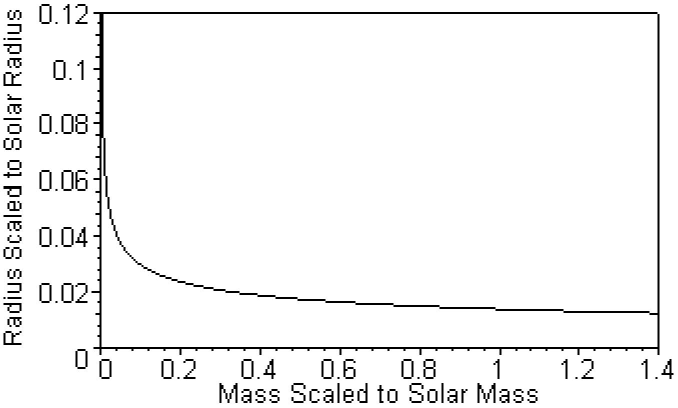

Figure 2. shows (40) scaled to the solar properties of Mass and Radius. It can be seen, as expected, that as the mass of the white dwarf increases the radius decreases, this is due to the Gravitational energy overtaking the energy of the degenerate electrons pressure. At the limit of 1.4 M_0 the Radius $R = 1.228_{10}^{-2}\ R_0 = 8.552_{10}^{6}$ m.

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[6] The Chandrasekhar limit is between 1.4 and 1.5 solar masses. R. Sexl (1979) approximates this limit to 1.5, and according to E. Chaisson (2002) and A.C. Phillips (1994) this limit is approximated to 1.4.

[7] See Appendix 4. for the derivation

[8] See A.C. Phillips pg.168, the approximation of $\rho_c/6$ comes from calculations based on the polytrope model

The study of the White Dwarf is a good beginning in the understanding of stellar objects. The simplicity of the White Dwarf has to do with the fact that the main mechanism that keeps it stable is the equilibrium between electron degeneracy and gravitational pressure. With relating these two energy sources S. Chandrasekhar was the first to determine the limitation of the properties of the White Dwarf. With this model one can continue with models of Neutron stars which are much more dense, smaller in radius and composed entirely of neutrons, and models of Black Holes which hold the greatest challenge as these objects are so dense and compact that not even light can escape them. The knowledge of the White Dwarf is especially important because as for our sun that is with in the limit will one day in millions of years become a White Dwarf, and eventually cool and become a Brown Dwarf just floating in the abyss of space.

Appendix

1. Constants;

Solar Mass $M_0 = 1.99_{10}^{30}$ kg
Solar Radius $R_0 = 6.96_{10}^{8}$ m
Proton mass (i.e. hydrogen mass) $m_H = 1.67_{10}^{-27}$ kg
Electron mass $m_e = 9.10_{10}^{-31}$ kg
Planks constant $h = 6.63_{10}^{-34}$ Js and $\hbar = h/2\pi$
Gravitational constant $G = 6.67_{10}^{-11}\ Nm^2/kg^2$
Speed of light $C = 3_{10}^{8}\ m/s^2$

2. Setting equations (13) and (19) equal to each other to find the limiting mass (Chandrasekhar mass limit)

$P_r = n_e^{4/3}h(3/8\pi)^{1/3}c/4 = (Y_e\rho_c/m_H)^{4/3}h(3/8\pi)^{1/3}(c/4)$ with (20) the electron degeneracy
$P_c \approx [\pi/36]^{1/3}GM^{2/3}\rho_c^{4/3}$ Hydrostatic equilibrium
$P_r \approx P_c$
$(Y_e\cancel{\rho_e}/m_H)^{4/3}h(3/8\pi)^{1/3}(c/4) \approx [\pi/36]^{1/3}GM^{2/3}\cancel{\rho_e}^{\cancel{4/3}}$
solving for $M = M_{CH}$
$M_{CH} = M_0\ [((Y_e/m_H)^{4/3}h(3/8\pi^2)^{1/3}c) / (4G))]^{3/2} / M_0 = 1.072\ M_0$

3. Integral identities

$\int x^m/(a^2+x^2)^n\,dx = \int x^{m-2}/(a^2+x^2)^{n-1}\,dx - a^2\int x^{m-2}/(a^2+x^2)^n\,dx$
$\int x^2/(a^2+x^2)^{1/2}\,dx = (x/2)(a^2+x^2)^{1/2} - (a^2/2)\ln(x+(a^2+x^2)^{1/2})+C$
$\int x^2(a^2+x^2)^{1/2}\,dx = (x/8)(a^2+2x^2)(a^2+x^2)^{1/2} - (a^4/8)\ln(x+(a^2+x^2)^{1/2})+C$

4. Solving for an approximation of the central density

Start with (36) and (9) and set them equal to each other
$P_c = 0.36\ GM^{2/3}\rho_c^{4/3}$
$P_{n\text{-}r} = (n_e^{5/3}\ h^2(3/8\pi)^{2/3})/ 5m$
$P_c = P_{n\text{-}r}$
$0.36\ GM^{2/3}\rho_c^{4/3} = ((Y_e\rho_c/m_H)^{5/3}\ h^2(3/8\pi)^{2/3})/ 5m$
ρ_c is easily solved for
$\rho_c = (0.36G5m_e/h^2)^3(m_H/Y_e)^5(8\pi/3)^2M^2 = 8.17_{10}^{-52}M^2$
$<\rho> = \rho_c/6$

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