Matrix Completion Problems

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Outline

1. Preliminaries
2. The Unspecified System Matrix
3. The Specified Basis Matrix
4. Connecting the Unspecified System and Specified Basis Matrices
5. The Linear Transformation $L : X \to AX - XA^T$
6. New Patterns
7. Conclusion

Definitions

Definition

A *partial matrix* B is a rectangular array with some entries specified where the remaining unspecified entries are free to be chosen.

Definition

A $n \times n$ *partial matrix pattern* $\beta = \{(i_1, j_1), \cdots, (i_k, j_k)\}$, $1 \leq i_t, j_t \leq n$, $t = \{a, \cdots, n\}$, is a set of specified entry locations.

Definition

Let $\beta \subseteq \{(i,j) | i,j \in \{1, \cdots, n\}\}$, then $B = (b_{ij})$ is a β-*partial matrix* if b_{ij} is specified and $b_{ij} \in \mathbb{F}$ if and only if $(i,j) \in \beta$.

Definition

A *completion* of a β-partial matrix $B = (b_{ij})$ is a matrix $\widehat{B} = (\widehat{b}_{ij})\ in M_n(\mathbb{F})$ in which $\widehat{b}_{ij} = b_{ij}$ whenever $(i, j) \in \beta$.

Definition

Given a matrix equation, a pattern is *admissible* if unspecified entries can always be completed so that the resulting matrix satisfies the given matrix equation.

Examples

- For the 4×4 case, this is the partial matrix pattern $\beta = \{(2,1),(2,3),(4,3),(3,4)\}$.

$$\begin{bmatrix} \square & \square & \square & \square \\ \# & \square & \# & \square \\ \square & \square & \square & \# \\ \square & \square & \# & \square \end{bmatrix}.$$

$\#$ is specified
$\square$ is unspecified

Examples

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- This partial matrix specifies the partial matrix pattern β.

$$\begin{bmatrix} \square & \square & \square & \square \\ 17 & \square & \pi & \square \\ \square & \square & \square & .5 \\ \square & \square & 273 & \square \end{bmatrix}.$$

$\square$ is unspecified

Examples

- For the 4×4 case, this is the partial matrix pattern $\beta = \{(2,1),(2,3),(4,3),(3,4)\}$.
- This partial matrix specifies the partial matrix pattern β.
- this matrix completes the partial matrix specifying β.

$$\begin{bmatrix} 0 & e & e^{\pi} & .257 \\ 17 & 11 & \pi & 10 \\ 1.33 & 9.87 & 1 & .5 \\ 53 & .001 & 19.63 & 8 \end{bmatrix}.$$

The Kronecker Product and Vec Operator

Lemma (HJ)

Let $A \in M_{mn}(\mathbb{F})$,$B \in M_{pq}(\mathbb{F})$, and $C \in M_{mq}(\mathbb{F})$ be given and let $X \in M_{np}(\mathbb{F})$ be unknown. The matrix equation

$$AXB = C$$

is equivalent to the system of qm equations in np unknowns given by

$$(B^T \otimes A)\mathrm{vec}(X) = \mathrm{vec}(C)$$

that is,

$$\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X)$$

Definition

We may use the previous lemma to define the following matrix.

Definition

Let $A_1, \ldots, A_k, B_1, \ldots, B_k, \in M_n(\mathbb{F})$. Let L be the linear transformation $L : X \to A_1XB_1 + \cdots + A_kXB_k$. Then the *unspecified system matrix* $\Phi(L)$, is defined as follows. $\Phi(L) = B_1^T \otimes A_1 + \ldots B_k^T \otimes A_k$.

Completing Matrices

Lemma

Let $A_1, \ldots, A_k, B_1, \ldots, B_k, C \in M_n(\mathbb{F})$ and let L be the linear transformation $L : X \to A_1XB_1 + \cdots + A_kXB_k$. Let $\alpha \subseteq \{(i,j) | i,j \in \{1, \ldots, n\}\}$. If $X = (x_{ij})$ is an $\alpha-$partial matrix, then there is a completion $\hat{X}$ of X with $A_1\hat{X}B_1 + \cdots + A_k\hat{X}B_k = C$ if and only if

$$\mathrm{vec}(C) - \sum_{(i,j)\in\alpha} x_{ij}\Phi(L)_{n(j-1)+i} \in span\{\Phi(L)_{n(j-1)+i} | (i,j) \in \alpha^C\}.$$

This lemma tells us when The matrix X may be completed and describes the system of equations that must be solved in order to do so.

Finding Admissible patterns

Corollary

For $A_1, \ldots, A_k, B_1, \ldots, B_k, C \in M_n(\mathbb{F})$, $L : X \to A_1XB_1 + \cdots + A_kXB_k$, and $\alpha \subseteq \{(i,j)|i,j \in \{1,\ldots,n\}\}$, the following statements are equivalent:

1. For any $\alpha -$ *partial* matrix X there exists a completion $\hat{X}$ such that $A_1\hat{X}B_1 + \cdots + A_k\hat{X}B_k = \mathbf{0}$
2. $\text{rank}(\Phi(L) = \text{rank}(\Phi(L)_{\alpha^c})$

This corollary tells us that the admissible patterns are those that set unspecified entries against columns of the unspecified system matrix such that these columns span the column space of the unspecified system matrix.

Definition

Definition

Let $L : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a linear transformation, and $\mathcal{B} = \{V_1, V_2, \ldots, V_n\}$ be a basis for the nullspace of L. The *specified basis matrix* is

$$\psi(\beta) = [\mathrm{vec}(\mathbf{V}_1), \mathrm{vec}(\mathbf{V}_2), \ldots, \mathrm{vec}(\mathbf{V}_n)].$$

Completing Matrices

Lemma

Let $L : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a linear transformation, $\mathcal{B}$ the basis of the nullspace of L, and $\alpha \subseteq \{(i,j) | i,j \in \{1,\ldots,n\}\}$ with $|\alpha| = k$. If X is an α-partial matrix, then there exists a completion $\widehat{X}$ of X such that $\widehat{X}$ is in the nullspace of L if and only if there exists $\mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{F}^k$ such that $\Psi_\alpha(\mathcal{B})\mathbf{c} = \begin{bmatrix} x_{i_1 j_1} \\ \vdots \\ x_{i_k j_k} \end{bmatrix}$. Moreover, in this event $\mathrm{vec}(\widehat{X}) = \Psi(\mathcal{B})c$.

This result tells us which matrices are completable.

Finding Admissible patterns

Lemma

Let $L : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a linear transformation, $\mathcal{B}$ be a basis of the nullspace of L, and $\alpha \subseteq \{(i,j) | i,j \in \{1,\ldots,n\}\}$ with $|\alpha| = |\mathcal{B}|$. Then any α-partial matrix X may be completed uniquely to be in the nullspace of L if and only if $\Psi_\alpha(\mathcal{B})$ is non-singular.

This result tells us which patterns are admissible.

Completing Matrices

Theorem

Let $L : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a linear transformation, $\mathcal{B}$ the basis of the nullspace of L, $\alpha \subseteq \{(i,j) | i,j \in \{1,\dots,n\}\}$ with $|\alpha| = k$, and $X = (x_{ij})$ be an α-partial matrix. Suppose $L = A_1XB_1 + \cdots + A_tXB_t$. The following statements are equivalent:

1. *There is a completion $\widehat{X}$ of X such that $\widehat{X}$ is in the nullspace of L.*
2. $\sum_{(i,j)\in\alpha} x_{ij}\Phi(L)_{n(j-1)+i} \in span\{\Phi(L)_{n(j-1)+i} | (i,j) \in \alpha^C\}$.
3. *There is a* $c = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{F}^k$ *such that* $\Psi_\alpha(\beta)\ c = \begin{bmatrix} x_{i_1j_1} \\ \vdots \\ x_{i_kj_k} \end{bmatrix}$.

This result tells us that the unspecified system matrix approach and the specified basis matrix approach will give the same results for completing matrices.

Finding Admissible Patterns

Theorem

Let $L : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a linear transformation, $\mathcal{B}$ the basis of the nullspace of L, $\alpha \subseteq \{(i,j)|i,j \in \{1,\ldots,n\}\}$ with $|\alpha| = |\mathcal{B}|$, and $X = (x_{ij})$ be an α-partial matrix. Suppose $L = A_1XB_1 + \cdots + A_kXB_k$. The following statements are equivalent:

1. *For any α-partial matrix X there is a completion $\widehat{X}$ of X such that $\widehat{X} \in null(L)$.*
2. $rank(\Phi(L) = rank(\Phi(L)_{\alpha^c})$.
3. $rank(\Psi_\alpha(\beta)) = |\mathcal{B}|$ *i.e.* $\Psi_\alpha(\beta)$ *is non-singular.*

This result tells us that the unspecified system matrix approach and the specified basis matrix approach will find the same patterns.

Some Tools

Definition

A *generalized eigenvector* of a matrix A is a nonzero vector $\overrightarrow{v}$ corresponding to an eigenvalue λ with algebraic multiplicity $k \geq 1$ such that $(\mathbf{A} - \lambda \mathbf{I})^{\mathbf{k}} \overrightarrow{v} = \mathbf{0}$.

1. $(\lambda_1 I - A)\mathbf{v_{11}} = \mathbf{0}$

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1. $(\lambda_1 I - A)\mathbf{v_{11}} = \mathbf{0}$
2. $(\lambda_1 I - A)\mathbf{v_{12}} = \mathbf{v_{11}}$

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3. $(\lambda_1 I - A)\mathbf{v_{13}} = \mathbf{v_{12}}$

Basis Theorem

Theorem

Let $A \in M_{m,m}(\mathbb{C})$, $B \in M_{n,n}(\mathbb{C})$, $X \in M_{m,n}(\mathbb{C})$. Let γ_i be an eigenvalue of A. Let $(\lambda - \gamma_i)^{p_i}$ be an elementary divisor of A. Let $\mathbf{a_{ir}}$ be the generalized eigenvectors of $-A$ associated with the eigenvalue $-\gamma_i$ with rank r, $r = 1, 2, \ldots p_i$ such that $(-A + \gamma_i I_m)\mathbf{a_{ir}} = \mathbf{a_{i,r-1}}$.
Similarly, let δ_j be and eigenvalue of B and let δ_j^* be an eigenvalue of B^*.
Let $(\lambda - \delta_i^*)^{q_j}$ be an elementary divisor of B^*. Let $\mathbf{b_{jq}}$ be the generalized eigenvectors of B^* associated with the eigenvalue δ_i^* with rank s, $s = 1, 2, \ldots q_j$ such that $(B^* + \delta_j^* I_n)\mathbf{b_{jq}} = \mathbf{b_{j,q-1}}$.
Let $L : X \to AX + XB$ be the linear transformation mapping $M_{m,n}(\mathbb{C})$ onto itself. Let $\mu_{ij} = min(p_i, q_j)$ and define

$$X_{ijk} = a_{ik} b_{j1}^* + a_{i,k-1} b_{j2}^* \cdots + a_{i1} b_{jk}^*$$

Then the set $\{X_{ijk} | \gamma_i + \delta_j = 0, k = 1, 2 \ldots \mu_{ij}\}$ forms a basis for the nullspace of L.

Basis Theorem

Theorem

Let $A \in M_n(\mathbb{R})$, $X \in M_n(\mathbb{C})$. Let γ_i be an eigenvalue of A. Let $(\lambda - \gamma_i)^{p_i}$ be an elementary divisor of A. Let $\mathbf{a_{ir}}$ be the generalized eigenvectors of $-A$ associated with the eigenvalue $-\gamma_i$ with rank r, $r = 1, 2, \ldots p_i$ such that $(-A + \gamma_i I_m)\mathbf{a_{ir}} = \mathbf{a_{i,r-1}}$.
Similarly, let δ_j be and eigenvalue of B and let δ_j^* be an eigenvalue of B^*. Let $(\lambda - \delta_i^*)^{q_j}$ be an elementary divisor of B^*. Let $\mathbf{b_{jq}}$ be the generalized eigenvectors of B^* associated with the eigenvalue δ_i^* with rank s, $s = 1, 2, \ldots q_j$ such that $(B^* + \delta_j^* I_n)\mathbf{b_{jq}} = \mathbf{b_{j,q-1}}$.
Let $L : X \to AX - XA^T$ be the linear transformation mapping $M_{m,n}(\mathbb{C})$ onto itself. Let $\mu_{ij} = min(p_i, q_j)$ and define

$$X_{ijk} = a_{ik} b_{j1}^* + a_{i,k-1} b_{j2}^* \cdots + a_{i1} b_{jk}^*$$

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Let $L : X \to AX - XA^T$ be the linear transformation mapping $M_{m,n}(\mathbb{C})$ onto itself. Let $\mu_{ij} = min(p_i, q_j)$ and define

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Let $A \in M_n(\mathbb{R})$, $X \in M_n(\mathbb{C})$. Let γ_i be an eigenvalue of A. Let $(\lambda - \gamma_i)^{p_i}$ be an elementary divisor of A. Let $\mathbf{a_{ir}}$ be the generalized eigenvectors of $-A$ associated with the eigenvalue $-\gamma_i$ with rank r, $r = 1, 2, \ldots p_i$ such that $(-A + \gamma_i I_m)\mathbf{a_{ir}} = \mathbf{a_{i,r-1}}$.
Let $L : X \to AX - XA^T$ be the linear transformation mapping $M_{m,n}(\mathbb{C})$ onto itself. Define

$$X_{ik} = a_{ik}a_{i1}^* + a_{i,k-1}a_{i2}^* \cdots + a_{i1}a_{ik}^*$$

Then the set $\{X_{ik} | k = 1, 2 \ldots p_i\}$ forms a basis for the nullspace of L.

Construction Example

Suppose that the matrix A has eigenvalues λ_1 and λ_2. Suppose that the multiplicity of λ_1 is 3 and generalized eigenvectors are v_{11}, v_{12}, v_{13}. In the same manner, λ_2 has multiplicity 1 and generalized eigenvector v_{21}.Then, according to the basis theorem:

1. $X_{11} = v_{11} v_{11}^T$
2. $X_{12} = v_{12} v_{11}^T + v_{11} v_{12}^T$
3. $X_{13} = v_{13} v_{11}^T + v_{12} v_{12}^T + v_{11} v_{13}^T$

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4. $X_{21} = v_{21} v_{21}^T$

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3. $X_{13} = v_{13}v_{11}^T + v_{12}v_{12}^T + v_{11}v_{13}^T$
4. $X_{21} = v_{21}v_{21}^T$
5. Basis: $\{X_{11}, X_{12}, X_{13}, X_{21}\}$

Permutation Similarity

Definition

Permutationally similar: $A \approx_p B$ if there exists a permutation matrix P such that $A = PBP^{-1}$.

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 4 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Then

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 4 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

is permutationally similar to A.

Finding Completions

Lemma

Let $A, B \in M_n$. Let $\sigma \in S_n$ with $P \in M_n$ being a representation of σ, and X, Y be partial matrices. If $A = PBP^{-1}$, $X = PYP^{-1}$ and there is a completion $\widehat{X}$ such that $A\widehat{X} - \widehat{X}A^T = \mathbf{0}$, then there exists a $\widehat{Y}$ such that $B\widehat{Y} - \widehat{Y}B^T = \mathbf{0}$.

Finding Patterns

Theorem

Let $A, B \in M_n$, $\sigma \in S_n$ with $P \in M_{n,n}$ being a representation of σ, and $A = PBP^{-1}$. Let $C = \{\alpha | \alpha$ is an admissible pattern for $A\}$, and let $D = \{\beta | \beta$ is an admissible pattern for $B\}$. Then $f : C \to D$ defined as $f(\alpha) = \{\sigma(i), \sigma^{-1}(j) | (i, j) \in \alpha\}$ is a bijection.

The Jordan Canonical Form

$$\mu_A(x) = (x - \alpha_1)^2 (x - \alpha_2)^3 (x - \alpha_3)$$

$$A = P \begin{bmatrix} \alpha_1 & 1 & 0 & 0 & 0 & \\ 0 & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_3 \end{bmatrix} P^{-1}$$

where $P = \begin{bmatrix} v_{11} & v_{12} & v_{21} & v_{22} & v_{23} & v_{31} \end{bmatrix}$

L-pattern Similarity

Definition

Suppose $\{\mathbf{v_1}, \cdots, \mathbf{v_k}\}$ is a basis of generalized eigenvectors of the matrix $A \in M_n(\mathbb{F})$ such that PAP^{-1} is in Jordan canonical form, where $P = [\mathbf{v_1}, \cdots, \mathbf{v_k}]$, with Jordan blocks $J_{n_1}(\lambda_1), \cdots, J_{n_k}(\lambda_k)$ ordered with respect to the basis. We then say that the matrix A is *L-pattern similar* to B, denoted $A \sim_L B$, if PBP^{-1} is in Jordan canonical form with Jordan blocks $J_{n_1}(\mu_1), \cdots, J_{n_k}(\mu_k)$ where $\lambda_i = \lambda_j \Rightarrow \mu_i = \mu_j$. If $\lambda_i = \lambda_j \Leftrightarrow \mu_i = \mu_j$, then A and B are said to be *strongly L-pattern similar*, denoted by $A \approx_L B$.

Example

$$A = P \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} P^{-1}, B = P \begin{bmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 1 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_2 \end{bmatrix} P^{-1},$$

$$\text{and } C = P \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 1 & 0 \\ 0 & 0 & \rho_2 & 0 \\ 0 & 0 & 0 & \rho_3 \end{bmatrix} P^{-1},$$

where $P = [\mathbf{v_{11}}, \mathbf{v_{21}}, \mathbf{v_{22}}, \mathbf{v_{31}}]$ and v_{ij} is the jth generalized eigenvector corresponding to the ith eigenvalue. Then A is L-pattern similar to B and strongly L-pattern similar to C.

Direct Results From Definition

Proposition

L-pattern similarity is a preorder.

Lemma

Let $A, B \in M_n(\mathbb{F})$ such that A is L-pattern similar to B. If α is an admissible pattern for A, then it will also be an admissible pattern for B.

Direct Results From Definition

Proposition

Strong L-pattern similarity is an equivalence relation.

Lemma

Let $A, B \in M_n(\mathbb{F})$ such that A is strongly L-pattern similar to B. Then the admissible patterns for A are also the admissible patterns for B.

Summary

1. The Unspecified Entry Approach
2. The Specified Entry Approach
3. Basis For Nullspace of $AX - XA^T$
4. New Patterns

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Any Questions, Comments, or Concerns?