ON GENERALIZED NASH RATIONALIZATION OF COLLECTIVE CHOICE FUNCTIONS

MIKHAIL FREER

George Mason University

Abstract. This paper analyzes collective outcomes in games from a revealed preference perspective. A collective choice function is *rationalizable* if there are "rational" individual preferences, such that the observed choices are the only equilibria. We consider a generalized concept of Nash equilibrium, which should be robust to deviation by both individuals and some exogenously given coalitions. The paper provides sufficient as well as necessary conditions for the collective choice function to be rationalizable given some notion of rationality. Likewise, we show that the conditions coincide and become a criteria if we relax the definition of equilibrium to the standard definition of Nash.

JEL codes: C72; C92
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1 Introduction

Revealed preference theory, established by Samuelson (1938), addresses the problem that although we can observe agents' choices, we may not be able to observe their preferences. This approach has been used to develop and apply tests for individual choice functions to be

E-mail address: (Freer) mfreer@gmu.edu.
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consistent with particular assumptions about preferences.[1] However, the literature on testable implication of the collective choice is sparser. Sprumont (2000) provides necessary and sufficient conditions for a collective choice function to be rationalizable as a Nash equilibrium in pure strategies, using transitivity and completeness as notion of rationality. Ray and Zhou (2001) addresses the question of rationalizability as a subgame perfect Nash equilibrium under transitive and complete preferences. Demuynck and Lauwers (2009) generalizes the Sprumont (2000) finding providing the criteria for rationalizability as Nash equilibrium in mixed strategies, assuming independence, transitivity and completeness as notion of rationality. Additionally, there are several papers studying the testable implications of the particular (classes of) games[2] and the papers investigating the rationalizability of the collective choice as the result of maximizing Pareto relation,[3] which are not directly related to the subject of this paper.

Nash equilibrium was initially criticized for its weakness as compared to cooperative solution concepts. The basis for the criticism was that Nash equilibrium is robust only for deviations by individual players. Aumann (1959) proposed the concept of *strong Nash equilibrium*, which requires every equilibrium to be robust to deviations of any coalition, and Bernheim et al. (1987) proposed *coalition-proof*

[1]Afriat (1967) and Richter (1966) proposed a test for the existence of a utility function that rationalizes observed behavior. Varian (1983) provides tests for homothetic rationalizability and the existence of an expected utility function that rationalizes preferences. Forges and Minelli (2009) provides a test for the existence of a concave utility function. Echenique and Saito (2015) provides a test for the existence of subjective expected utility. Chambers and Echenique (2016) contains the systematic overview of the revealed preference results.

[2]For instance, Cherchye et al. (2013) and Carvajal et al. (2013) study the empirical implications of Cournot competition, Lee (2012) investigates testable implications of equilibrium behavior in zero-sum games.

[3]For instance, Sprumont (2001) studies continuous sets of actions and Echenique and Ivanov (2011) studies finite sets of actions. Both papers address the two-person case.

Nash equilibrium, which requires every equilibrium to be robust to the deviations of any sustainable coalition. Both of those concepts are to some extent extreme, as they allow every set of players to form a coalition. Myerson (1977) proposed a more general idea of the games with coalitional structure. The set of coalitions which can be formed by players is exogenous, and only those coalitions can deviate. We follow this approach and consider equilibrium which is robust for a deviation by any coalition from a given set of coalitions.

Standard notion of rationality (includes transitivity, completeness, independence) was criticized based on experimental evidence on observed behavior in the context of individual decision making as well as in game theoretic settings. Since Allais (1953), various researchers have been challenging the independence axiom.[4] Furthermore, there are numerous alternative theories of behavior under risk (e.g., Kahneman and Tversky (1979), Quiggin (1982), Cerreia-Vioglio et al. (2015)). Another assumption which is usually criticized is selfish behavior, i.e., monotonicity with respect to the monetary payoffs. In certain types of games, this assumption can be violated due to the fact that one may care about other players' payoffs (e.g. Charness and Rabin (2002)), inequality (e.g. Fehr and Schmidt (1999), Bolton and Ockenfels (2000)) etc. The generalized version of rationality we use allows for any (testable) competing theories as the notion of rationality.

This paper provides sufficient conditions as well as necessary conditions for the collective choice function to be rationalizable as a coalitional Nash equilibrium (given the coalitional structure) under our generalized version of rationality. Moreover, we show that if we relax the definition of equilibrium to the Nash equilibrium (keeping the generalized notion of rationality), then the necessary and sufficient conditions coincide. However, we show that Nash rationalization is neither necessary nor sufficient for the coalitional Nash rationalization.

[4]See Ellsberg (1961), Kahneman and Tversky (1979),Battalio et al. (1985) and Holt (1986) for experimental designs and results.

The remainder of this paper is organized as follows. Section 2 provides necessary definitions. Section 3 presents results of the paper. Section 4 discusses the results. Section 5 concludes. All proofs not in the text are collected in the Appendix.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ be the set of players. For every $j \in N$ let X_j be the set of possible strategies. Let $X = \times_{j \in N} X_j$ be the set of possible strategy profiles (joint outcomes). Let $\mathcal{G} \subseteq \{G \subseteq X : \exists \varnothing \neq Y_j \subseteq X_j \text{ and } G = \times_{j \in N} Y_j \neq \varnothing\}$ be the set of non-empty Cartesian product sets induced by X, which are the observed games.

Let $\mathcal{K} \subseteq 2^N$, such that $N \subseteq \mathcal{K}$ be the non-empty set of admissible coalitions. Denote by G_K a projection of G, that is if $G = \times_{j \in N} Y_j$, then $G_K = \times_{j \in K} Y_j$. Denote by $G_K^x = G_K \times x_{-K}$, that is a subgame of the game G in which the strategies of $N \backslash K$ are fixed. A set of observed games $\mathcal{G}$ satisfies **domain restriction** if there are $G_K^x \in \mathcal{G}$ for every $G \in \mathcal{G}$, for every $K \in \mathcal{K}$ and for every $x \in G$.

A collective choice function $C : \mathcal{G} \to 2^X$ assigns to every element G of $\mathcal{G}$ a non-empty set $C(G) \subseteq G$. A collective choice function $C : \mathcal{G} \to 2^X$ is **$\mathcal{K}$-noncooperative** when for every $G \in \mathcal{G}$, $x \in C(G)$ if and only if $x \in C(G_K^x)$ for every $K \in \mathcal{K}$ and $G_K^x \subseteq G$. Further we use term noncooperative for N-noncooperative collective choice functions. Note that $\mathcal{K}$-noncooperativeness is a synonym of the "consistency" condition, the latter is a property of (coalition-proof) Nash equilibrium (see Peleg and Tijs (1996)).

Figure 1 illustrates the assumption of $\mathcal{K}$-noncooperative behavior. Here and further Player 1 chooses rows, Player 2 – columns and Player 3 – matrices. Asterisk symbol denotes the chosen cell. DRS is chosen from the G, and therefore it has to be chosen by every individual (see cases (b) $G_{\{2\}}^{DRS}$, (c) $G_{\{1\}}^{DRS}$ and (e) $G_{\{3\}}^{DRS}$). Moreover, since players 1 and 2 can form a coalition, there is an additional subgame $G_{\{1,2\}}^{DRS}$ and DRS should be chosen from that subgame as well.

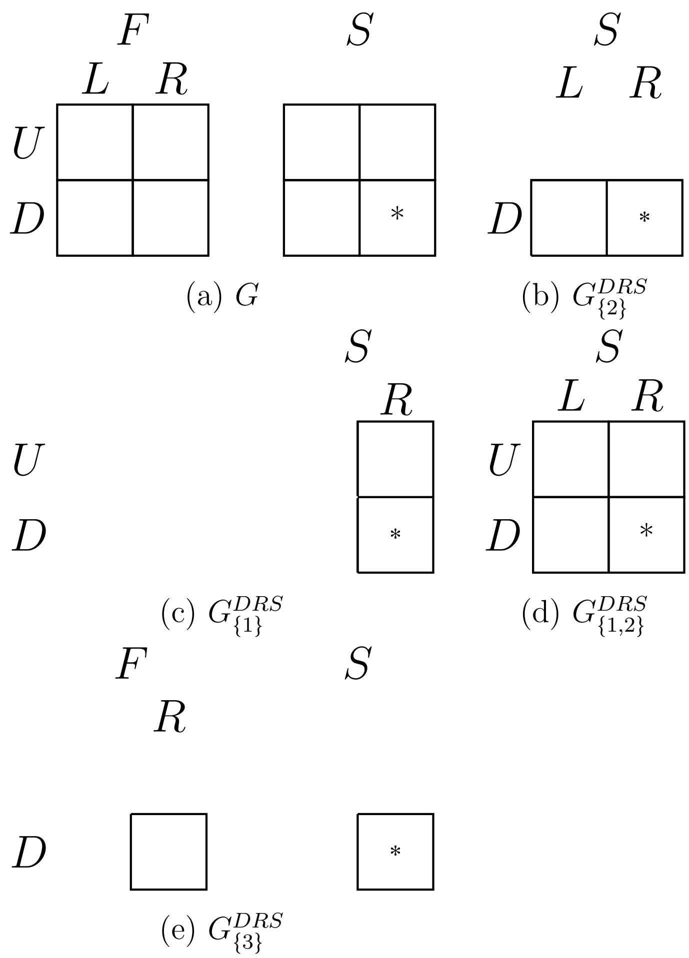


FIGURE 1. $\mathcal{K}$-noncooperative collective choice function, for coalitional structure $\mathcal{K} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$

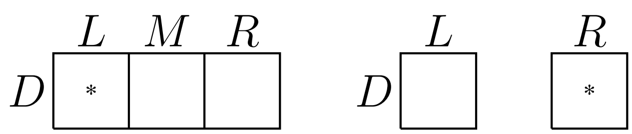
Note that Figure 1 illustrates only the "forward" implication of $\mathcal{K}$-noncooperative behavior, that is if x is chosen in a game G, then it should be chosen by every G^x_K. However, it is also necessary that if y is not chosen, then there is at least one $K \in \mathcal{K}$, such that y is not chosen from G^y_K. For instance, ULF, there is at least one player or coalition that prefers not to choose ULF from G^{ULF}_K.

2.1 (Revealed) Preferences A set $R \subseteq X \times X$ is said to be a preference relation. We denote the set of all preference relations on X by $\mathcal{R}$. We denote the inverse relation $R^{-1} = \{(x, y) | (y, x) \in R\}$.

We denote the symmetric (indifferent) part of R by $I(R) = R \cap R^{-1}$ and the asymmetric (strict) part by $P(R) = R \backslash I(R)$. We denote the incomparable part by $N(R) = X \times X \backslash (R \cup R^{-1})$. A preference relation R is **complete** if $(x, y) \in R \cup R^{-1}$ for all $x, y \in X$ (or equivalently $N(R) = \varnothing$). A preference relation R is **transitive** if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

Definition 1. *A preference relation R' is an **extension** of R, denoted $R \preceq R'$, if $R \subseteq R'$ and $P(R) \subseteq P(R')$.*

Every collective choice function generates a revealed preference relation. Let $(x, y) \in R_K(G_K^x)$ if and only if $x \in C(G_K^x)$ and $y \in G_K^x$. Denote by $R_K = \bigcup_{G_K^x \in \mathcal{G}} R_K(G_K^x)$. Denote by $\bar{R}_K = \bigcup_{K' \subseteq K} R_{K'}$. Denote by $R_{K_j^*} = \bigcup_{K \in \mathcal{K}:\ j \in K} \bar{R}_K$ for every $j \in N$. A relation R_K satisfies **internal consistency** if and only if $R_K(G_K^x) \preceq R_K$ for every $G_K^x \in \mathcal{G}$.



(a) G (b) $\tilde{G}$

Figure 2. Violation of Internal Consistency

Figure 2 presents the violations of internal consistency. Player chooses DL from the game G and DR from the game $\tilde{G}$. Then, $(DL, DR) \in P(R_1(G))$ and $(DR, DL) \in P(R_1(\tilde{G}))$. Therefore, $(DL, DR) \in I(R_1)$, hence, $P(R_1(G)) \nsubseteq R_1$. This implies that R_1 can not be an extension of $R_1(G)$.

2.2 Notion of Rationality We use functions over preference relations to impose the notion of rationality. The simplest example of such a function is the *transitive closure*, which adds (x, z) to R, whenever there is a finite sequence $x = y_1, \ldots, y_n = z$, such that R contains (y_j, y_{j+1}) for every $j = 1, \ldots, n-1$. The transitive closure allows every preference relation which can be extended by its transitive closure to have a complete and transitive extension (see Richter (1966)).

Definition 2. *A function $F : \mathcal{R} \to \mathcal{R}$ is said to be*

- ***monotone*** *if for all $R, R' \in \mathcal{R}$, if $R \subseteq R'$, then $F(R) \subseteq F(R')$,*
- ***closed*** *if for all $R \in \mathcal{R}$, $R \subseteq F(R)$,*
- ***idempotent*** *if for all $R \in \mathcal{R}$, $F(F(R)) = F(R)$,*
- ***algebraic*** *if for all $R \in \mathcal{R}$ and all $(x, y) \in F(R)$, there is a finite relation $R' \subseteq R$ such that $(x, y) \in F(R')$,*
- ***weakly expansive*** *if for any $R = F(R)$ and $N(R) \neq \varnothing$, there is a nonempty set $S \subseteq N(R)$ such that $R \cup S \preceq F(R \cup S)$.*

Any function $F : \mathcal{R} \to \mathcal{R}$ that is monotone, closed and idempotent is called a **closure**. A closure is algebraic as defined above if any element of the closure can be obtained from applying the closure to a finite subset of the original relation.[5] Weak expansiveness impose conditions on the fixed points of F.[6] In particular, it guarantees that for every fixed point of F there is a set of non-comparable pairs (comparisons) which can be added to the fixed point, such that the enlarged relation can be extended by F. Demuynck (2009) shows that if F is a weakly expansive algebraic closure, then existence of a complete fixed point extension of preference relation is equivalent to the fact that this preference relation can be extended by F.

As we previously mentioned, the idea behind F is to impose the desired properties or the "notion of rationality". Further we assume that every notion of rationality includes *transitivity and completeness* of preference relations. A function $F : \mathcal{R} \to \mathcal{R}$ **induces transitivity** if $T(F(R)) = F(R)$. That is, every fixed point of F is also a fixed point of T, and as it was shown by Demuynck (2009) every fixed point of T is a transitive relation.

2.3 Equilibrium Given a relation R on a set X and a subset $G \subseteq X$, we denote by $M(R, G) = \{x \in G | \forall y \in G, (y, x) \notin P(R)\}$ the set of **maximal elements** of G according to the relation R. Let R_j for

[5]See e.g. Davey and Priestley (2002), definition 7.12.

[6]Fixed point of F is such R, that $F(R) = R$.

$j \in N$ be individual preference relations of players. Denote by Π_K the **Pareto relation**, $(x, y) \in \Pi_K$ if and only if $(x, y) \in R_j$ for all $j \in N$. Equivalently Pareto relation can be defined as $\Pi_K = \bigcap_{j \in K} R_j$.

The set of maximal elements can be defined for any preference relation (complete or not), therefore, it can be defined for Pareto relation as well. Note that for every singleton coalition Pareto relation is equal to the preference relation of the player. Before we proceed note that game can be defined as a tuple (G, R), where $G \in \mathcal{G}$ and $R = (R_1, \ldots, R_n)$ is preference profile.

Definition 3. *A joint outcome $x \in X$ is* ***$\mathcal{K}$-Nash equilibrium*** *of (G, R^*) if and only if $x \in M(\Pi^*_K, G^x_K)$ for every $K \in \mathcal{K}$.*

$\mathcal{K}$-Nash equilibrium has two special cases, one of which is Nash equilibrium (only coalitions are players on their own) and strong Nash equilibrium (every subset of players is a coalition). Further we refer to the N-Nash equilibrium as to Nash equilibrium.

Definition 4. *A collective choice function $C : \mathcal{G} \to 2^X$ is* ***$(F, \mathcal{K})$-Nash Rationalizable*** *when there is $R^*_j = F(R^*_j)$ for every $j \in N$, such that $\forall G \in \mathcal{G}$, $x \in C(G)$ if and only if x is $\mathcal{K}$-Nash equilibrium of (G, R^*).*[7]

Further we refer to the (F, N)-Nash rationalizability as to F-Nash rationalizability.

3 Results

Before we present the results, let us state the assumptions which are persistently made in every statement. We assume F to be a weakly expansive algebraic closure that induces transitivity. Moreover, we assume that $\mathcal{G}$ satisfies domain restriction. Both assumptions do not

[7]We assume the common notion of rationality. Formally we can assign every player individual F_j according to which she should rational, and all the further results hold under that assumption. However, we keep the assumption of the common notion of rationality, because individual notions of rationality seem artificial.

relate to the collective choice function. Rather, they specify the properties of the domain of the observed games and the properties of the notion of rationality.

Proposition 1 (Sufficient Conditions)**.** *If*

- C *is* $\mathcal{K}$*-noncooperative,*
- R_K *satisfies internal consistency for every* $K \in \mathcal{K}$*,*
- $R_K \preceq R_{K_j^*}$ *for every* $K \in \mathcal{K}$*,* $K \subseteq K_j^*$ *and*
- $R_{K_j^*} \preceq F(R_{K_j^*})$ *for every* $j \in N$*.*

Then, $C : \mathcal{G} \to 2^X$ *is* $(F, \mathcal{K})$*-Nash Rationalizable*

The idea of the proof is to assign every player the preference relation that is an extension of $F(R_{K_j^*})$. Demuynck (2009) shown that there is a complete fixed point extension of such relation which is an extension of $F(R_{K_j^*})$, which is an extension of R_j (by transitivity of $\preceq$). Therefore, the set of maximal elements would coincide with the observed choices function at the individual level. Maximal elements of the Pareto relation (based on the completed individual preference relations) would coincide with the observed choices at the coalitional level, since every individual preference relation takes into account the revealed preferences of coalitions.

Proposition 2 (Necessary Conditions)**.** *If* $C : \mathcal{G} \to 2^X$ *is* F $\mathcal{K}$*-Nash Rationalizable, then*

- C *is* $\mathcal{K}$*-noncooperative*
- R_j *satisfies internal consistency for every* $j \in N$ *,*
- $R_j \preceq R_{K_j^*}$ *for every* $j \in N$*, and*
- $R_j \preceq F(R_j)$ *for every* $j \in N$*.*

Necessary and sufficient conditions have similar structure, however, they are different except the first one. $\mathcal{K}$-noncooperative behavior stands for the noncooperative nature of decision making. Internal consistency is the condition which guarantees that there is some complete preference relation which drives all the observed choices. Moreover, we require individual preference relations to be consistent with the coalitional preferences. The major difference is in the last condition, which requires a revealed preference relations to be extendable by F. Recall that this is equivalent to the existence of complete fixed point extension of the preference relation. It is obviously necessary to guarantee the existence of the complete extension of the individual revealed preference relations; however, it does not have to be true for the coalitional preferences.

3.1 Partial Rationalizability Our definition of rationalizability requires not only that all the chosen points are equilibria, but also that all non-chosen points are not equilibria. Note that the latter heavily relies on domain restriction. Therefore, if we want to relax the domain restriction, we must also relax the rationalizability concept. A collective choice function $C : \mathcal{G} \to 2^X$ is said to be **partially** $(F, \mathcal{K})$**-Nash rationalizable** when there is $R_j^* = F(R_j^*)$ for every $j \in N$, such that $\forall G \in \mathcal{G}$, if $x \in C(G)$ then x is $\mathcal{K}$-Nash equilibrium of (G, R^*).

Remark 1 (Partial $(F, \mathcal{K})$-Nash Rationalizability)**.** *Every* $C : \mathcal{G} \to 2^X$ *is partially* $(F, \mathcal{K})$*-Nash rationalizable.*

To show this let us assume that $R_j = X \times X$ for every $j \in N$. Hence, every player is indifferent between every pair of joint outcomes, and all of them are $\mathcal{K}$-Nash equilibria. Therefore, every outcome of the joint choice function is $\mathcal{K}$-Nash equilibrium as well.

Remark 1 shows that partial rationalizability does not have empirical content. This result is in line with the finding in Sprumont (2000), that every collective choice function is partially Nash rationalizable if we take complete and transitive preferences as a notion of rationality.

3.2 F-Nash Rationalizability F-Nash rationalizability is the extreme case under which $\mathcal{K} = N$ and is a generalization of the Nash rationalizability for an arbitrary notion of rationality. The only allowed coalitions in this case are $K = \{j\}$ for some $j \in N$. Hence, necessary (Proposition 2) and sufficient (Proposition 1) conditions become equivalent, since $R_{K_j^*} = R_j$.

Corollary 1 (Nash Rationalizability). *$C : \mathcal{G} \to 2^X$ is F-Nash rationalizable if and only if*

- *C is noncooperative,*
- *R_j satisfies internal consistency for every $j \in N$, and*
- *$R_j \preceq F(R_j)$ for every $j \in N$.*

Corollary 1 is a generalization of the results from Sprumont (2000) and Demuynck and Lauwers (2009). It first assumes only finite sets of alternatives and uses transitivity and completeness as a notion of rationality. Secondly, it assumes sets of alternatives to be the mixture space over the finite set of alternatives and uses transitivity, independence and completeness as a notion of rationality.

4 Discussion

We address three major points in this discussion. First, we show that sufficient conditions are not tight, i.e, we provide an example of $(F, \mathcal{K})$-Nash rationalizable collective choice function that fails sufficient conditions. Second, we show that F-Nash rationalizability is not sufficient for $(F, \mathcal{K})$-Nash rationalizability of the collective choice function. Third, we show that $(F, \mathcal{K})$-Nash rationalizability is not sufficient for the F-Nash rationalizability of the collective choice function.

Figure 3 illustrates the case of rationalizable collective choice function that violates internal consistency of R_K. This follows from the fact that $(UL, DR) \in P(R(G))$, because UL is chosen and DR is not and at the same time $(UL, DR) \in I(R(\tilde{G}))$ because both points are chosen. However, the underlying Pareto ordering can contain $(CM, DR) \in P(\Pi^*_{\{12\}})$ and $\{(UL, DR), (UL, CM)\} \subset N(\Pi^*_{\{12\}})$. Figure 4 shows the possible distribution of payoff (in utils) in the matrix, such that there

	L	C	R
U	*		
M		*	
D			

(a) G

	L	R
U	*	
D		*

(b) $\tilde{G}$

FIGURE 3. $(F, \mathcal{K})$-Nash rationalizable collective choice function, which violates internal consistency of R_K. $\mathcal{K} = \{\{1\}.\{2\}, \{1,2\}\}$

are complete, transitive and monotone individual preference relation which rationalize collective choice function.

	L	C	R
U	$10, 5$	$0, 0$	$0, 0$
M	$4, 9$	$5, 10$	$4, 9$
D	$0, 0$	$4, 9$	$3, 8$

FIGURE 4. Payoff matrix for rationalization of the collective choice function from Figure 3

By construction of R_K every point that is chosen is better than every point that is not chosen, although this does not have to be true in general. Therefore, formally there may be such coalitional preferences that every non-chosen point is dominated at least by one chosen point, and at least one of those relations must be internally consistent. There are many of such "candidate relations", hence, the condition that includes them would not be easily testable.[8]

[8]This becomes a dimension problem, which was proven to be infeasible to solve even in the case of the finite amount of alternatives if there are more than two players (see Yannakakis (1982)).

However, there is a consistency condition for R_K which is necessary for the rationalization of collective choice function. We also use that condition to show that $(F, \mathcal{K})$-Nash rationalizability has an empirical content comparing to F-Nash rationalizability. That is, there is an F-Nash rationalizable function, which is not $(F, \mathcal{K})$-Nash rationalizable.

Remark 2. *If $C : \mathcal{G} \to 2^X$ be an $(F, \mathcal{K})$-Nash Rationalizable collective choice function, then, for every $K \in \mathcal{K}$, $P^{-1}(\bigcap_{j \in K} F(R_j)) \cap \bar{R}_K = \varnothing$.*

The condition stated in the remark is the type of extension condition, but we cannot claim that the $\bar{R}_K$ is an extension of $\bigcap_{j \in K} F(R_j)$, as formally the latter does not have to be a subset of $\bar{R}_K$.

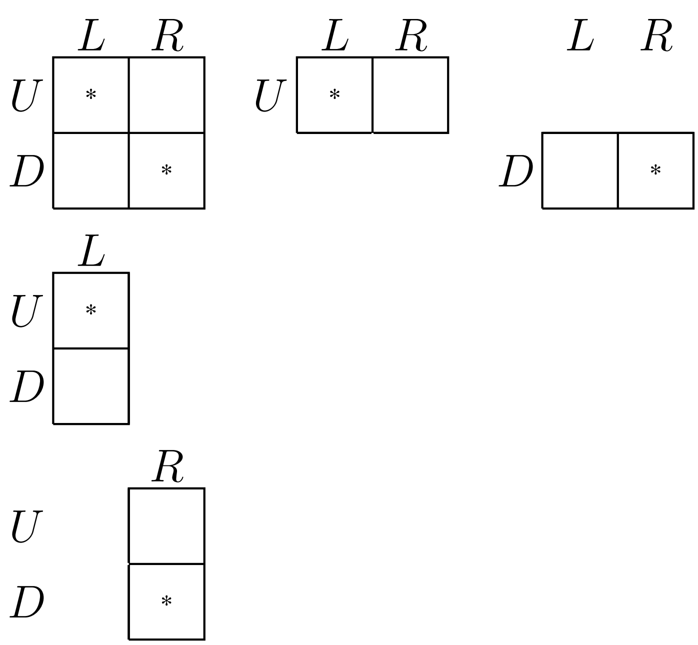


FIGURE 5. Collective choice function which is F-Nash rationalizable, but not $(F, \mathcal{K})$-Nash rationalizable. $\mathcal{K} = \{\{1\}.\{2\}, \{1, 2\}\}$

Consider collective choice function from Figure 5 and function F, such that $F(R) = R \cup \{(UL, DR)\}$. Obviously, F is a weakly expansive algebraic closure. Collective choice function does not violate conditions from Corollary 1, therefore, it is F-Nash rationalizable. However, if $\{1,2\} \in \mathcal{K}$, then this collective choice function is not $(F,\mathcal{K})$-Nash rationalizable, as it contains a violation of the Remark 2. Note that $F(R_1) = \{(UL, DL), (DR, UR), (UL, DR)\}$ and $F(R_2) = \{(UL, UR), (DR, DL), (UL, DR)\}$, hence, $P(F(R_1) \cap F(R_2)) = \{(UL, DR)\}$. While $R_{\{1,2\}}$ contains both (UL, DR) and (DR, UL), as both points are chosen and therefore, equivalent.

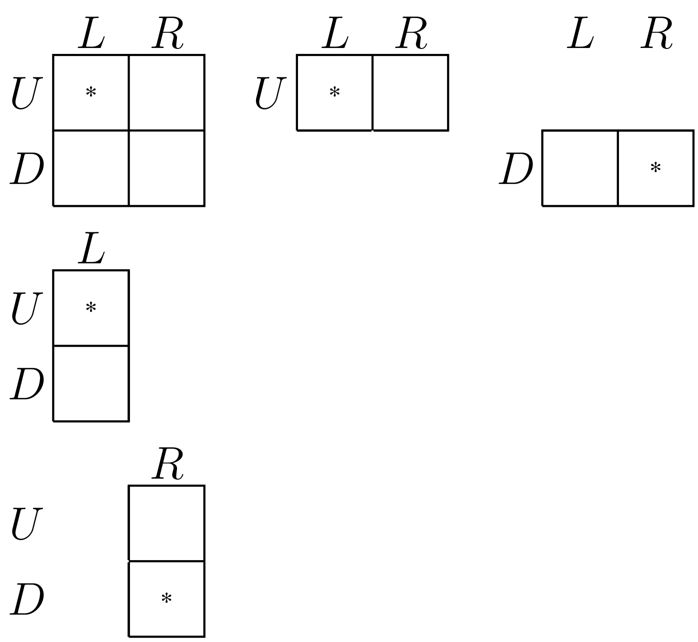


FIGURE 6. Collective choice function which is $(F,\mathcal{K})$-Nash rationalizable, but not F-Nash rationalizable. $\mathcal{K} = \{\{1\}.\{2\}, \{1,2\}\}$

Figure 6 illustrates the example of the $(F,\mathcal{K})$-Nash rationalizable collective choice function which is not F-Nash rationalizable. Note that DR is chosen in both $G^{DR}_{\{1\}}$ and $G^{DR}_{\{2\}}$; however, it is not chosen by the coalition $\{1,2\}$. Thus, DR cannot be $\mathcal{K}$-Nash equilibrium in the game, even though it is obviously the Nash equilibrium. Therefore, this function is not F-Nash rationalizable, due to the fact it fails the noncooperativeness condition.

5 Concluding Remarks

We can bring some "economic flavour" into the rationalization concept by using the generalized notion of equilibrium. One can think that preferences should respect some partial order, e.g., monotonicity in payoffs (selfishness), first order stochastic dominance, or inequality aversion. Any property which can be expressed as a partial order can be simply incorporated into F (see Demuynck (2009)) and, therefore, included in the notion of rationality.

Note that, we consider only complete information games. We also assume beliefs to be rational, and consistent with the infinite regression of rationality, i.e, "A knows that B knows that A knows ... that A is rational". Relaxation of any of those assumptions opens a wide range of interesting questions.

Appendix

Proposition 3 (Theorem 2 from Demuynck (2009))**.** *Let F be a weakly expansive algebraic closure. There is a complete extension of $R^* = F(R^*)$ of a preference relation R if and only if $R \preceq F(R)$.*

Proof can be found in Demuynck (2009).

Lemma 1. *Let $R \subseteq R'$, then $R \preceq R'$ if and only if $P^{-1}(R) \cap R' = \varnothing$.*

Proof can be found in Freer and Martinelli (2016).

Lemma 2. *Let $\mathcal{A}$ be a finite collection of sets. Let R be a preference relation, so that $R \preceq R_A$ for every $A \in \mathcal{A}$. Then $R \preceq \bigcup_{A \in \mathcal{A}} R_A$.*

Proof. Obviously $R \subseteq \bigcup_{A \in \mathcal{A}} R_K$. On the contrary assume the contrary, there is $(y, x) \in P(R)$ such that $(x, y) \in \bigcup_{A \in \mathcal{A}} R_A$. Then $(x, y) \in R_T$ for some $T \in \mathcal{A}$. But then R does not extend R_A. $\square$

Observation 1. *For every $i \neq j$ $P^{-1}(R_i) \cap R_j = \varnothing$.*

Proof. This follows from the simple fact, that $G_j^x \cup G_i^x = \{x\}$. Recall that R_i and R_j contain pairs (x, y), and the only pair they can contain is $(x, x) \in I(R)$ if x was chosen. □

Lemma 2 and Observation 1 allow us to prove the following lemma.

Lemma 3. *If $P^{-1}(R_j) \cap R_K = \varnothing$ for every $j \in K \in \mathcal{K}$, then $R_j \preceq R_{K_j^*}$.*

Proof. Assume on the contrary that this is not true, then $(y, x) \in P(R)$ and $(x, y) \in \bar{R}_K$. Recall that $\bar{R}_K = \bigcup_{K' \subseteq K} R_K$, then $(x, y) \in R'_K$ for some $K' \in \mathcal{K}$ by construction of $\bar{R}_K$. For every non-singleton coalition it cannot be true by assumption of the Lemma. For every singleton $i \neq j$ it cannot be true by Observation 1. Then we can conclude the proof by applying Lemma 2. □

Proof of Proposition 1

Proof of Proposition 1. By definition of $(F, \mathcal{K})$-Nash rationalizability, there exist $R_j^* = F(R_j^*)$ for every $j \in N$, such that x is a $\mathcal{K}$-Nash equilibrium if and only if $x \in M(G_K^x, \Pi_K^*)$ for every $K \in \mathcal{K}$. The proof is organized as follows,

- (1) We construct complete extensions of R_j, which are fixed points of F,
- (2) We show that according to those extensions x is $\mathcal{K}$-Nash equilibrium if and only if it is chosen from G:
  - (2.1) We show that every chosen joint outcome is $\mathcal{K}$-Nash equilibrium, i.e., a maximal element in G_K^x for every $K \in \mathcal{K}$,
  - (2.2) We show that every element which is not chosen cannot be a $\mathcal{K}$-Nash equilibrium.

(1) Note that we guaranteed that R_j is internally consistent and $R_j \preceq R_{K_j^*} \preceq F(R_{K_j^*})$. Hence, Proposition 3 implies that there is a complete fixed point extension R_j^* such that $F(R_{K_j^*}) \preceq R_j^* = F(R_j^*)$.[9] Then, by transitivity of $\preceq$ we can conclude that $R_j \preceq R_j^*$.

(2.1) By $\mathcal{K}$-noncooperativeness $x \in C(G)$ if and only if $x \in C(G_K^x)$ for every $K \in \mathcal{K}$. Hence, $y \in G_K^x : (x, y) \in R_K \subseteq F(R_{K_j^*}) \subseteq R_j^*$ for all $j \in K$. Then, $(x, y) \in \Pi_K^*$, this implies that there is no $(y, x) \in P(\Pi_K^*)$, because P represents the asymmetric part of relation.

(2.2) Let $x \in G \backslash C(G)$. By domain restriction and $\mathcal{K}$-noncooperativeness there is G_K^x, such that $x \notin C(G_K^x)$. Therefore, there is $y \in G_K^x \subseteq G$, such that $(y, x) \in P(R_K(G_k^x)) \subseteq P(R_K) \subseteq P(R_{K_j^*}) \subseteq P(R_j^*)$, for every $j \in K$. Then, $(y, x) \in P(\Pi_K^*)$, that implies, that $x \notin M(G_K^x, \Pi_K^*)$. □

Proof of Proposition 2

Lemma 4. *If $C : \mathcal{G} \to 2^X$ is $(F, \mathcal{K})$-Nash rationalizable then $R_j \preceq R_j^*$.*

Proof. First let us show that $R_j \subseteq R_j^*$. Take $(x, y) \in R_j$, then x was chosen in some game in which y was present. Hence, $x \in C(G_j^x) = M(G_j^x, R_j^*)$, since x is a maximal element there is no y which is strictly preferred to it, then by completeness of R_j^* $(x, y) \in R_j^*$.

Now let us show that $P(R_j) \subseteq P(R_j^*)$. Take $(x, y) \in P(R_j)$, then for G_j^x $x \in C(G_j^x)$ and $y \notin C(G_j^x)$, hence, x is a maximal element i G_j^x, according to R_j^*. Then, by completeness and transitivity of R_j^* for every $y \in G_j^x \backslash C(G_j^x)$, $(x, y) \in P(R_j^*)$.[10] □

Proof of Proposition 2. Suppose a collective choice function $C : \mathcal{G} \to 2^X$ is $(F, \mathcal{K})$-Nash rationalizable. First note that C is $\mathcal{K}$-noncooperative. For all $x \in C(G)$ no coalition has a profitable deviation, otherwise $x \notin M(\Pi_K^*, G)$. If $x \notin C(G)$, then x is not $\mathcal{K}$-Nash equilibrium and there is $K \in \mathcal{K}$, such that $x \notin M(G_K^x, \Pi_K^x)$. This implies, that x can not be $\mathcal{K}$-Nash equilibrium of $G_K^x \in \mathcal{G}$ and therefore, cannot be chosen from G_K^x.

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[9]Recall that F is an idempotent function, therefore, $F(R)$ is trivially F consistent. This extension is an extension of R_j by transitivity of $\preceq$ relation.

[10]In this case completeness guarantees that $(x, y) \in R_j^*$ and transitivity guarantees that $(x, y) \notin I(R_j^*)$, because otherwise y has to be maximal point as well.

Assume R_j is not internally consistent. Then, by Lemma 1 there is G_j^x such that $(y,x) \in P(R(G_j^y))$ and $(x,y) \in R_j$. According to Lemma 4 $(x,y) \in I(R_j^*)$. Hence, completeness of R_j^* implies, that if y is a maximal element in G_j^y, then x has to be maximal element as well. Since C is $(F,\mathcal{K})$-Nash rationalizable, then all maximal elements has to be chosen, i.e. $x \in C(G_j^y)$, this implies, that $(x,y) \in R_j(G_j^y)$.

This part is proven by Lemma 3, hence we need to show that $P^{-1}(R_j) \cap R_K = \varnothing$ for every $j \in K \in \mathcal{K}$ Assume to the contrary that $(x,y) \in P^{-1}(R_j) \cap R_K$ for some $j \in K \in \mathcal{K}$. This implies, that there is $G_K^x \in \mathcal{G}$ such that $x \in C(G_K^x)$, then by $\mathcal{K}$-noncooperativeness $x \in C(G_j^x)$ for every $j \in K$. At the same time $(y,x) \in P(R_j)$ implies, that $y \in C(G_j'^y)$ and $x \in G_j'^x$. This implies that y can be obtained from x by individual deviation. Recall that $y \in G_K^x$, then $y \in G_j^x$, hence, $(x,y) \in R_j$. This implies that $(y,x) \notin P(R_j)$.

F-consistency of R_j follows from Proposition 3 and Lemma 4, since F is a weakly expansive algebraic closure, if R_j is not F-consistent then there is no fixed point extension of it. $\square$

Proof of Remark 2

Lemma 5. *If $C : \mathcal{G} \to 2^X$ is $(F,\mathcal{K})$-Nash Rationalizatible, then $P^{-1}(\Pi_K^*) \cap R_K = \varnothing$.*

Proof. Assume on the contrary, that there is $(y,x) \in P(\Pi_K^*)$ and $(x,y) \in R_K$. The latter implies, that there is G_K^x, such that $x \in C(G_K^x)$ and $y \in G_K^x$. Recall that Nash rationalizability requires all chosen points of G_K^x to be maximal points of Π_K^x, therefore, $(y,x) \notin P(\Pi_K^*)$. $\square$

Note that $\Pi_K^* \preceq \Pi_{K'}^*$ for every $K, K' \in \mathcal{K}$, such that $K \subseteq K'$. This follows from the fact that Pareto relation is constructed as the intersection of individual preference relations. Then, Lemma 5 also implies, that $P^{-1}(\Pi_K^*) \cap \bar{R}_K = \varnothing$, because $P^{-1}(\Pi_K^*) \cap R_{K'} = \varnothing$ for every $K' \subseteq K$.

Proof of Remark 2. Recall that Lemma 4 guarantees that if C is $(F, \mathcal{K})$-Nash Rationalizable, then $R_j \preceq R_j^*$ for every $j \in N$. Moreover, monotonicity of F and the fact that $R_j^* = F(R_j^*)$ implies, that $F(R_j) \preceq R_j^*$. Therefore, $\bigcap_{j \in K} F(R_j) \preceq \Pi_K^* = \bigcap_{j \in K} R_j^*$ for every $K \in \mathcal{K}$. Since we have shown that $P^{-1}(\Pi_K^*) \cap \bar{R}_K = \varnothing$. Then, $P^{-1}(\bigcap_{j \in K} F(R_j)) \cap \bar{R}_K = \varnothing$, since $P(\bigcap_{j \in K} F(R_j)) \subseteq P(\Pi_K^*)$. $\square$

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