A Revealed Preference Test of Quasi-Linear Preferences

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Abstract

We provide criteria for a set of observed choices to be generated by a quasi-linear preference relation. These criteria are valid if choice sets are compact and downward closed and do not require preferences to be convex. We show that under the added assumption that choice sets can be represented by linear budget sets, our condition also implies the existence of a quasi-linear *utility representation* of preferences. We implement a test of quasi-linear preferences using both experimental and panel survey data. Using experimental data, we find that while subjects are generally consistent with the generalized axiom of revealed preferences they are no closer to quasi-linearity in money than a subject choosing at random would be. Using panel survey data we find partial support for the assumption of quasi-linearity in one of the goods.

1 Introduction

We provide criteria for a set of observed choices over compact and downward closed sets to be generated by a quasi-linear preference relation. When these conditions are satisfied there exists a complete, transitive, monotone and quasi-linear preference relation consistent with observed behavior. In the special case in which budget sets are linear, our condition implies the existence of a concave and continuous utility function that rationalizes the data.

It is difficult to overstate the importance of the assumption of quasilinear preferences in both the theoretical and empirical literature in economics. It plays a crucial role in implementation theory, the theory of the household as well as applied welfare analysis. For instance, it is a necessary assumption for the Revenue Equivalence theorem (Krishna, 2009; Myerson, 1981), the existence of truth-revealing dominant strategy mechanism for public goods (Green and Laffont, 1977) and the Rotten Kids theorem (Becker, 1974; Bergstrom and

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Cornes, 1983). It is also an often invoked assumption in applied welfare analysis (Domencich and McFadden, 1975; Allcott and Taubinsky, 2015). Our paper is concerned with the empirical content of this assumption.

These above-mentioned applications differ greatly on the nature of the choice sets faced by agents and are many times silent about whether preferences are convex or not. A main advantage of our approach is that it does not rely in either the convexity of preferences or the linearity of budget sets. Our test is built on the following observation. If preferences are quasi-linear, say in good x, it must be true that if $(x, y) \sim (x', y')$ then $(x+\alpha, y) \sim (x'+\alpha, y')$ for all α. This property can be directly tested and requires making no assumptions on the shape of the utility function (or existence of) or ancillary assumptions on the choice set. The added generality of our test makes it possible to device tests of quasi-linearity of preferences in strategic environments.

Brown and Calsamiglia (2007) provide a test for the existence of a quasi-linear, concave and monotone utility function that rationalizes choices over linear budget sets.[1] Relaxing these two assumptions implies that while we can guarantee the existence of a quasi-linear preference relation consistent with observed data, we cannot guarantee the existence of a quasi-linear utility function representing them.[2] We show, however, that our test is equivalent to Brown and Calsamiglia (2007) if choice sets a linear. This result is equivalent to Afriat (1967) who shows that if budgets are linear, convexity of preferences has no empirical content.[3]

The second contribution of the paper is to provide an empirical test of the quasi-linearity of preferences both using lab and field data. In our experiment, we mimic different consumption groups by offering gifts cards at discounted prices. We test if subjects preferences over 5 alternative goods in 30 different budgets can be rationalized by a quasi-linear preference ordering. We find that while support for the Generalized Axiom of Revealed Preferences is strong, the support for quasi-linearity of preferences is weak.[4] We conduct a second test of quasi-linearity of preferences using the Spanish Continuous Family Expenditure Survey (see

[1]More recently, Cherchye et al. (2015) proposed a test of generalized quasi-linear preferences (Bergstrom and Cornes, 1983). Their test is a generalization of Brown and Calsamiglia (2007) and also requires linear budget sets.

[2]We note that previous tests of quasi-linear preferences can be extended to more general budget sets if the assumption of concavity is maintained. In this case, the existence of a concave utility representation of preferences will be equivalent to the existence of a set of prices that contain observed choice sets and for which observed choices satisfy cyclical monotonicity.

[3]Convexity is also a crucial assumption in the test of separability of preferences, homotheticity (Varian, 1983a), expected utility theory (Varian, 1983b), preferences with habit formation (Crawford, 2010), collective model of household consumption (Cherchye et al., 2007) and subjective expected utility (Echenique and Saito, 2015).

[4]The test was designed to test quasi-linearity of preferences in money. We find that quasi-linearity of preferences fail even if we relax this requirement.

Beatty and Crawford (2011)).[5] In this dataset, we find stronger support of the assumption of quasi-linearity of preferences against the alternative of random choice. Due to the low variability in prices and income in this dataset, however, we conclude that further tests are needed to establish the empirical validity of the assumption of quasi-linear preferences.

Varian (1982) shows how to derive non-parametric bounds on welfare measures based solely on the satisfaction of the Generalized Axiom of Revealed Preferences. These bounds can be quite wide and uninformative (Hausman and Newey, 1995). We show that these bounds can be significantly narrowed under the assumption of quasi-linear preferences. This suggests that aggregate welfare measures can be derived without making assumptions about the unobserved heterogeneity of preferences.

The remainder of this paper is organized as follows. Section 2 provides the intuition for the proof of the necessary and sufficient conditions for the data to be consistent with the maximization of a preference ordering that is complete, monotone, transitive and quasi-linear. Section 3 presents the empirical results obtained by taking the test to the data. Section 4 discusses benefits of assuming that preferences are quasi-linear versus just assuming that preferences are rational. Section 5 presents the connection between our test and one from Brown and Calsamiglia (2007). Section 6 summarizes the results of the paper.

2 Theoretical Framework

Let us start by defining the quasi-linearity of preferences. Preferences are said to be *quasi-linear in the i-th component* if x being better than y implies that a shift of x along the i-th axis ($z = x + \alpha e_i$) is better then the same shift of y along the same axis ($w = y + \alpha e_i$). Wherer, x, y, w and z are consumption bundles and e_i is the i-th unit vector, i.e. $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the i-th place.

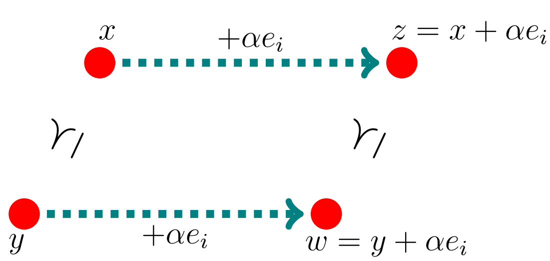


Figure 1: Quasilinear Preferences

Figure 1 shows this shift graphically. The dashed lines arrows represent shifts. If they are of the same length, then preferences are quasi-linear if x is better than y implies that

[5] Crawford (2010) aggregates the data into 14 categorie. We use a further aggregation into 5 categories as in Beatty and Crawford (2011).

$z = x + \alpha e_i$ is better than $w = y + \alpha e_i$.

2.1 Idea of the Proof

Let us show the intuition for the necessary condition on quasi-linearity of preferences and later we formally prove that it is sufficient as well. Figure 2 shows preferences that are inconsistent with quasi-linearity. It is enough to have x strictly better than y and $w = y+\alpha e_i$ weakly better than $z = x + \alpha e_i$ for some $\alpha \in \mathbb{R}$.

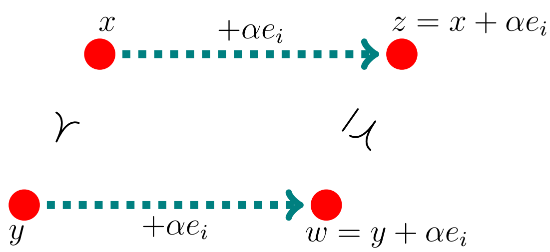


Figure 2: Contradiction of Quasi-linearity

Thus, if the revealed preference relation contains two pairs as in Figure 2, then the preference relation contains a contradiction of quasi-linearity. Clearly it is a necessary condition for quasi-linearity of preferences.

We intend to apply our test to a revealed preference relation derived from choice data. Hence, let us translate the case of violation of quasi-linearity to the context of revealed preferences. Let $(x^t, B^t)_{t=1,\ldots,T}$ be the finite consumption experiments, where B^t are budgets and x^t are chosen points.

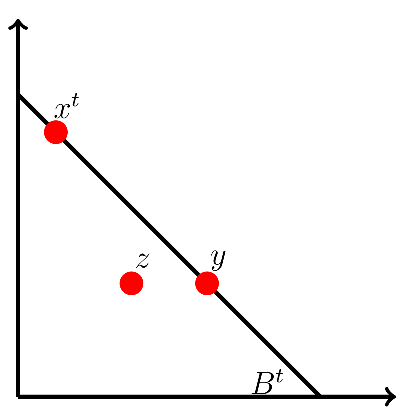


Figure 3: Revealed Preference Relation

Now we will define a revealed preference relation. In Figure 3, bundle x^t is better than y if y lies in budget B^t (including the boundary of the budget) and x^t is strictly better than z if z lies strictly inside (in the interior) of budget B^t. Note that the only bundles that are revealed better than others are the chosen bundles (x^t). That is, we do not know if y is better than z. Hence, to get a contradiction of quasi-linearity we need to consider at least two chosen points.

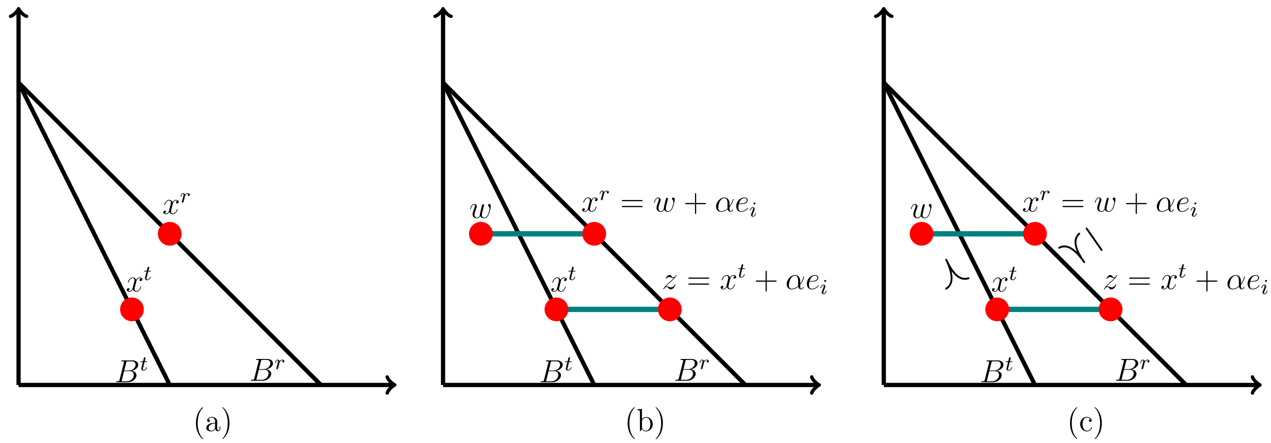


Figure 4: Constructing Test for Quasi-linearity

Figure 4(a) shows that bundle x^r is strictly better than the chosen bundle x^t. Let us explicitly show that this example contains the violation of quasi-linearity. Figure 4(b) shows that w is a shift[6] of the point x^r along the horizontal axis by the same amount as z is a shift of the point x^t. And from the definition of the revealed preference relation we know that x^t is strictly better than w, and x^r is strictly better than z. Figure 4(c) contains the case similar to one on Figure 2, thus the revealed preference relation violates quasi-linearity.

Figure 5(a) shows the condition necessary to eliminate the possibility of violations of quasi-linearity shown at the Figure 4(c). If the line from x^t (the strictly worse bundle) to z, [such that is still in the budget B^r (still less preferred than x^r)] is shorter than a line from x^r to the point w, [that lies in the budget B^r (is worse than x^t)], then there is a violation of quasi-linearity.

The blue line is the maximum distance (along the horizontal axis) from x^t to such point z that is still worse than x^t. The green line is the minimum distance from x^r to the y that is worse than x^t. If blue line is longer than green one then there is z' that is strictly worse than x^t and y' that is strictly worse than x^r. And these points can be equal shifts of x^r and x^t respectively. It is exactly the case shown in Figures 2 and 4(c). Hence, the test is simply checking that for any x^t that is strictly worse than x^r blue line must be shorter than the green one.

Since we would like to construct a preference relation that is transitive and quasi-linear, the test should take into account transitivity as well. We do this in the similar fashion as

[6]Note that we are not restricting the space to the positive orthant. Hence, the budget line (more general hyperplane) simply separates the space into two subspaces, and interior of budget is the subspace that lies "below" the budget line.

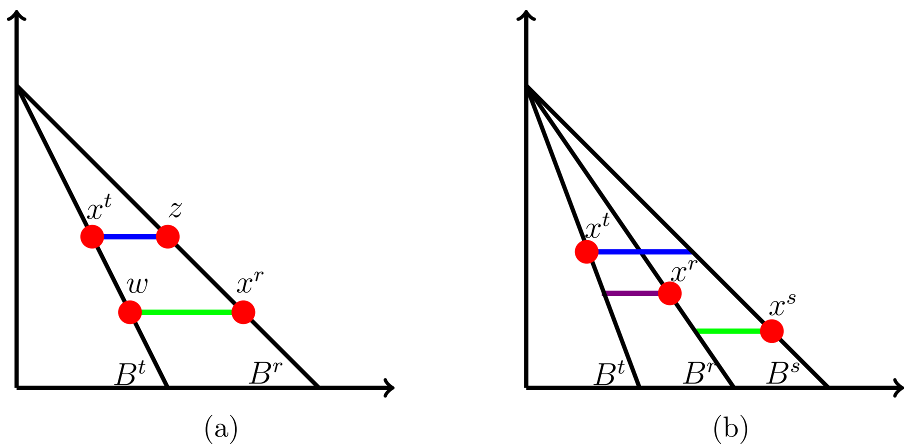


Figure 5: Test for Quasi-Linearity

the transformation of Weak Axiom of Revealed Preferences (WARP) into General Axiom of Revealed Preference (GARP). Hence, consider a chain of elements such that each is less preferred than a previous one and is achieved by shifting some chosen point. Sum of shifts of elements in the chain should be less than the shift from the last to the first one. Let us illustrate this with a simple example. Suppose we have three budgets and three chosen points x^t, x^r and x^s as illustrated in Figure 5(b). Then, x^s is better than both x^r and x^t and x^r is better than x^t. Figure 5(b) illustrates how the test works, it is passed if sum of lengths of green and purple lines is greater than length of the blue one.

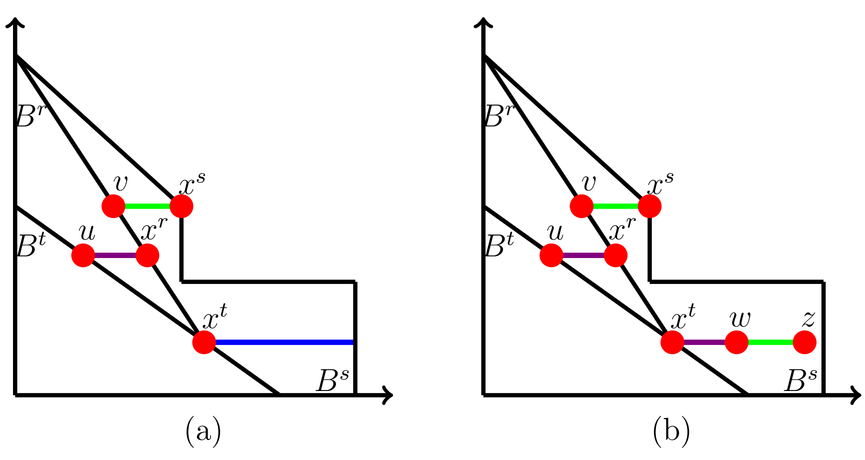


Figure 6: Violation of Quasi-linearity under Assumption of Transitivity

Let us show that the test shown on the Figure 5(a) is not capable of detecting violation of the joint hypothesis of quasi-linearity and transitivity, and we have to use use the case represented on the Figure 5(b) using the simple example. Figure 6(a) shows an example of a relation that passes the test of quasi-linearity (Figure 5(a)) but fails the joint test of quasi-linearity and transitivity (Figure 5(b)). Therefore, this set of choices cannot be generated by complete, transitive and quasi-linear preference relation. Distance from x^s to budget set B^t is less than distance from x^t to B^s, hence, there is no direct violation of quasi-linearity. Bundle x^t is preferred to u, so by quasi-linearity w is better than x^r and by transitivity w is better than v. Hence, applying quasi-linearity again we get that z is better than x^s, and at the same time x^s is strictly better than z, since z lies strictly inside of the budget set B^s. That is a violation of joint hypothesis of quasi-linearity and transitivity.

2.2 Test

Consider a set of alternatives $X \subseteq \mathbb{R}^N$. A set $R \subseteq X \times X$ is said to be a **preference relation**. We denote the set of all preference relations on X by $\mathcal{R}$. Denote the reverse preference relation by $R^{-1} = \{(x,y)|(y,x) \in R\}$, the symmetric part of R by $I(R) = R \cap R^{-1}$ and the asymmetric part by $P(R) = R \setminus I(R)$. Denote the non-comparable part by $N(R) = X \times X \setminus (R \cup R^{-1})$. A relation R' is said to be an **extension** of R, denoted by $R \preceq R'$ if $R \subseteq R'$ and $P(R) \subseteq P(R')$.

Definition 1. *Preference relation is said to be:*

Complete *if* $N(R) = \emptyset$;

Transitive *if for any* $x, y, z \in X$: $(x,y) \in R$ *and* $(y,z) \in R$ *implies* $(x,z) \in R$;

Monotone *if for any* $x \gg y$ $(x,y) \in R$;

Quasi-linear in the i-th component *if for any* $(x,y) \in R$ *and any* $\alpha \in \mathbb{R}$ $(x + \alpha e_i, y + \alpha e_i) \in R$.

Let $(x^t, B^t)_{t=1...T}$ be the **finite consumption experiment** where x^t are chosen points and B^t are budgets. We assume all budgets to be compact and monotone.[7] Denote by R_v the **revealed preference** relation,[8] that is $(x^t, y) \in R_v$ if $y \in B^t$, $(x^t, x^t) \in I(R_v)$ and $(x,y) \in P(R_v)$ for any $y \in B^t \setminus \{x^t\}$. Denote by $T(R_v)$ the **transitive closure** of the relation, i.e. $(x,y) \in T(R_v)$ if there is a sequence $S = s_1, s_2, ..., s_n$ such that for any $j = 1...n-1$ $(s_j, s_{j+1}) \in R_v$. Note that $(x,y) \in P(T(R_v))$ if $(x,y) \in T(R_v)$ and there is $k = 1...n-1$ such that $(s_k, s_k + 1) \in P_v$. Denote by $C = \{x^1, x^2, ..., x^T\}$ the *set of all chosen points* in the finite consumption experiment $(x^t; B^t)_{t=1..T}$. A revealed preference relation is said to be **acyclic** if it satisfies SARP.[9]

[7] $x \in B^t$, then any $y \leq x$ is also in B^t. And since we work on $\mathbb{R}^N$ it will also include elements with negative coordinates.

[8] All the results below can be achieved for weak rationalization as well, but the conditions will be inelegant.

[9] The consumption experiment $E = (x_i, B_i)_{i=1}^n$ satisfies the Strong Axiom of Revealed Preference (**SARP**) if for every integer $m \leq n$ and every $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$, $x_{i_{j+1}} \in B_{i_j}$ for $j = 1, \dots, m-1$ implies $x_{i_1} = x_{i_m}$ or $x_{i_1} \notin B_{i_m}$.

Definition 2. *A revealed preference relation satisfies* ***QLSARP*** *with respect to the i-th component if for any sequence of distinct elements* $x^{k_1}, ..., x^{k_n} \in C$ *and* $(\alpha, \beta_3, \ldots, \beta_n) \in \mathbb{R} \times \mathbb{R}_{++} \times \ldots \times \mathbb{R}_{++}$*, such that* $(x^{k_1}, x^{k_2} - \alpha e_i) \in P(T(R_v))$ *and* $(x^{k_j}, x^{k_{j+1}} - \beta_{j+1} e_i) \in T(R_v)$ *for* $j = 2, \ldots, n-1$*, then* $(x^{k_n}, x^{k_1} + (\alpha + \sum_{j=3}^{n} \beta_j) e_i) \notin T(R_v)$*.*

QLSARP is simply the formal statement of the test from Figure 5. Note that α does not need to be positive, i.e. (x^{k_1}, x^{k_2}) may be an element of R_v, while none of further pairs $(x^{k_j}, x^{k_{j+1}})$ can be in the revealed preference relation. Given a sequence of chosen points $x^{k_1}, \ldots, x^{k_n}$, $(\alpha, \beta_3, \ldots, \beta_n)$ are shifts (to the left) of the chosen points generating a new sequence, such that each point in this new sequence is preferred to the next one. QLSARP fails if and only if there is a sequence of chosen points and vector of such shifts, such that the point obtained by shifting the initial chosen point (to the right) by $(\alpha + \sum_{j=3}^{n} \beta_j)$ is less preferred than the last point of the sequence of chosen points. Note that if the sequence consists of x^{k_1}, x^{k_2} only, then we need to check that if $(x^{k_1}, x^{k_2} - \alpha e_i) \in P(T(R_v))$, then $(x^{k_2}, x^{k_1} + \alpha e_i) \notin T(R_v)$, which exactly coincides with Figure 5(a).

Theorem 1. *An acyclic revealed preference relation* R_v *generated by a finite consumption experiment with monotone and compact budgets has an extension that is complete, transitive, monotone and quasilinear in the i-th component if and only if* R_v *satisfies QLSARP with respect to the i-th component.*

Proof of Theorem 1 is in Appendix A and the proof of the similar result for the case of weak rationalization is presented in Appendix B. The pseudo-code algorithmic implementation of QLSARP is presented in Appendix C.

Note that Theorem 1 does not guarantee the existence of a utility function. If we assume that the set of alternatives is a subset of $\mathbb{Q}^N$, then existence of utility would follow immediately since the space is countable and any complete and transitive relation over no more than countable set of alternatives can be represented by a utility function.[10] However, the utility function is not necessarily continuous.

3 Testing Quasi-Linearity

We test quasi-linearity in two contexts: quasi-linearity in goods[11] and quasi-linearity in money. By quasi-linearity in goods we mean the quasi-linearity in at least one of goods. It is usually assumed that if consumer has to make choices among large variety of goods, then

[10] The proof for existence of utility representation of a preference relation over no more than countable set of alternatives can be found in Fishburn (1988). Moreover, the condition on the set of alternatives can be relaxed to being subset of $\mathbb{R}^{N-1} \times \mathbb{Q}$, using the result from Freer and Martinelli (2016).

[11] We relax the assumption and allow the goods to be different for different subjects. The detailed discussion on the assumptions of common versus individual numeraires is in Appendix.

his preferences are quasi-linear at least in one of them. Quasi-linearity in money is usually assumed, since money is a natural numeraire in which one can express the value of every good.

In order to test the assumption we have to use some benchmark. Our benchmark is rationality of preferences that is equivalent to the revealed preference relation to be consistent with GARP (see Varian (1982)). It is important to consider that people make mistakes. Some people may therefore not pass QLSARP exactly even though their underlying preferences are quasi-linear. For the measure of distance from rationality we use the Critical Cost Efficiency Index (CCEI) introduced by Afriat (1973). Since X is a linear space and B^t are simply subsets of linear space, we can introduce $B^t(e) = \{x \in X : \frac{x}{e} \in B^t\}$. Then, $R_v(e)$ is a revealed preference relation generated by the finite consumption experiment $(x^t, B^t(e))$. The CCEI for QLSARP can be defined as the maximum $e \in (0, 1]$ such that $R_v(e)$ satisfies QLSARP.[12]

Changing e changes the probability that a set of random choices will pass QLSARP. To control this we use the **predictive success index** introduced by Selten (1991). The predictive success index is defined as the difference between share of people that satisfies axiom at the given e and the probability that uniform random choices will satisfy the axiom at the same e. This index ranges between -1 and 1, with -1 meaning no subject passes even though random choice would with probability one and 1 meaning every subjects passes even though random choice never would. To estimate the probability that uniform random choices will satisfy the axiom we use the Monte Carlo method with 1000 simulated random agents for each set of prices.

3.1 Quasi-Linearity in Goods

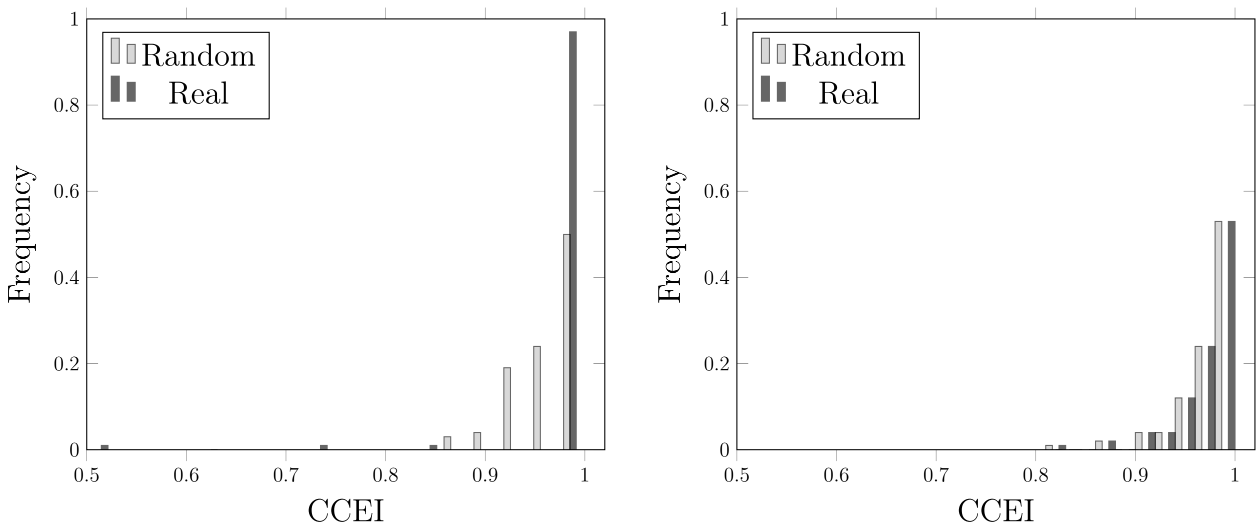
First, we test quasi-linearity of preferences in goods that is existence of the numeraire good. For this purpose we use data from Mattei (2000). It is experiment with 8 real consumption goods.[13] Each subject faced 20 different budgets and one (uniformly randomly chosen) consumer choice was implemented. The experiment was conducted with 20 economics and 100 business students from the University of Lausanne.[14] The payment was in goods and the average monetary equivalent of payment is \$30.5.

Figure 7 shows the distribution of CCEIs for GARP and QLSARP. Each consumer faced 8 goods, so, there will be 8 different CCEIs depending on the good in which we assume preference to be quasi-linear in. The CCEI for QLSARP is maximum of these eight different

[12] The CCEI can be defined similarly for any other axiom, e.g. GARP.

[13] The goods were: milk chocolate, biscuits, orange juice, iced tea, writing pads, plastic folders, diskettes, 'post it'

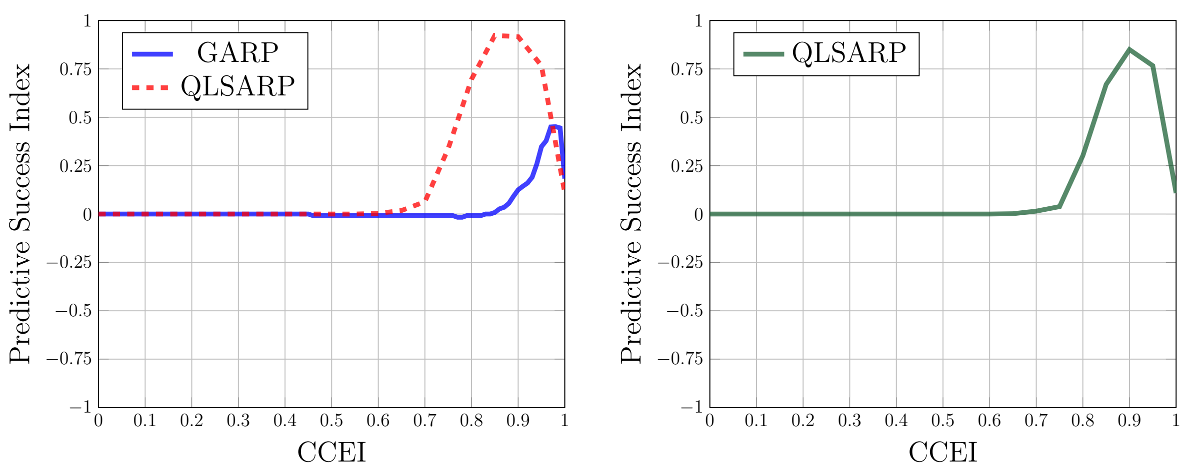
[14] The paper also contains similar non-incentivized expenditure survey for 320 real consumers. We do not include the results from non-incentivized treatment.



GARP QLSARP

Figure 7: CCEI distributions for GARP and QLSARP

CCEIs for each consumer. The CCEI levels for QLSARP are lower than ones for GARP. This is predictable, since the QLSARP is a stricter test. Let us then consider the Predictive Success Index for these two axioms.



(a) H_0: Random Choices (b) H_0: Random Choices that Pass GARP

Figure 8: Predictive Success Index

Figure 8 shows that quasi-linearity of preferences can not be rejected immediately. Figure 8a shows the predictive success index for QLSARP and GARP separately. It shows that at the low level of decision making error (high level of CCEI $\approx$ .95) GARP outperforms QLSARP (predictive success index for GARP is higher than one for QLSARP). Hence the hypothesis of rational preferences is more favorable rather than a hypothesis of quasi-linear preferences. While at the higher level of decision making error (lower level of CCEI $\approx .85$) QLSARP outperforms GARP.

Figure 8b shows the predictive success index of QLSARP conditioning on random choices that pass GARP, that is predictive success index equals 1 if none of the random choices that passed GARP pass QLSARP, while all real observations pass QLSARP; and predictive success index equals -1 if all random choices pass that pass GARP pass QLSARP as well, while none of real observations pass QLSARP. Note that difference between figures is small, since the original test for GARP is weak (see Figure 7). The idea of conditioning QLSARP Predictive Success Index on GARP, is that assume there is a person (random set of choices) that looks consistent with GARP (has rational preferences), what is the probability that this person will look consistent with QLSARP (has quasi-linear preferences) as well. Figure 8b shows that conditioning of QLSARP on random choices that pass GARP does not change the picture a lot. QLSARP still performs good enough at the lower level at CCEI ($\approx .9$).

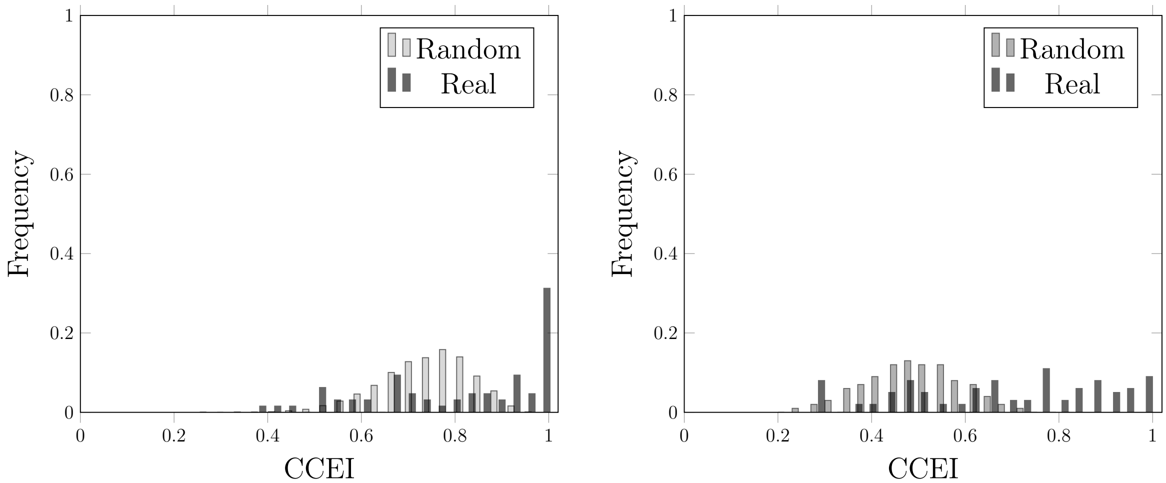
Figure 8 shows the trade-off between assuming rational and quasi-linear preferences. Agents are consistent with the assumption of rational preferences under the lower decision making error (higher CCEI). Assuming only rationality of preferences we gather very vague predictions of future behavior. While assuming quasi-linear preferences allows us to tighten these predictions significantly. However, assuming quasi-linear preferences requires assuming that decision-making error is higher.

3.2 Quasi-Linearity in Money

We now test quasi-linearity in money which is not possible to do using the data from Mattei (2000). To test quasi-linearity in money, we conduct an experiment[15] in which each subject had to allocate an endowment among five goods: Cash, Fandango Gift Card, Barnes and Noble gift card, Gap gift card and Mason Money. Each good stands for the category of goods and services on which subjects are expected to spend money. Fandango is movie theater ticket distributor and stands for entertainment spending. Barnes and Noble is book distributor and stands for necessities since textbooks are a required expenditure for students. Gap is clothing store and stands for durable goods. Mason Money is George Mason internal monetary system that can be used at any on-campus restaurant, therefore, it stands for food spending. The commodities are chosen as to minimize the transaction costs of consumption. The unit of measurement of each commodity is \$1. Subjects are asked to allocated a 100 tokens between the above described goods facing prices that are denominated in tokens per dollar. Each subject faces 30 decision problems, one of which is chosen at random to be

[15] For extensive description of experimental procedure, instructions and screen-shots see Appendix D.

implemented. The experiment was conducted with 64 George Mason undergraduates.[16]



(a) GARP

(b) QLSARP

Figure 9: CCEI distributions for GARP and QLSARP

Figure 9 shows distribution of GARP and QLSARP (in money) CCEIs for the experimental data. Figure 9a shows distribution of GARP CCEIs for randomly simulated choices[17] and actual choices. Figure 9a shows that distribution of CCEIs for real choices is shifted to the right in comparison to the distribution of random choices: mean GARP CCEI for real choices is .81 and for random choices is .75.[18] Figure 9b shows distribution of QLSARP CCEIs for randomly simulated choices and actual choices. The distribution of the QLSARP CCEIs for actual choices is shifted to the right and more dispersed than the distribution of QLSARP CCEIs for random choices. The mean QLSARP CCEI is .69 for actual choices and .49 for random choices.[19]

These results are similar if we compare actual choices to the choices of a "synthetic" subject. A "synthetic" subject is a collection of 30 budgets and associated decisions taken at random from all the menus produced by the experiment.[20]

Figure 10a shows the predictive success indexes for GARP and QLSARP. Figure 10b

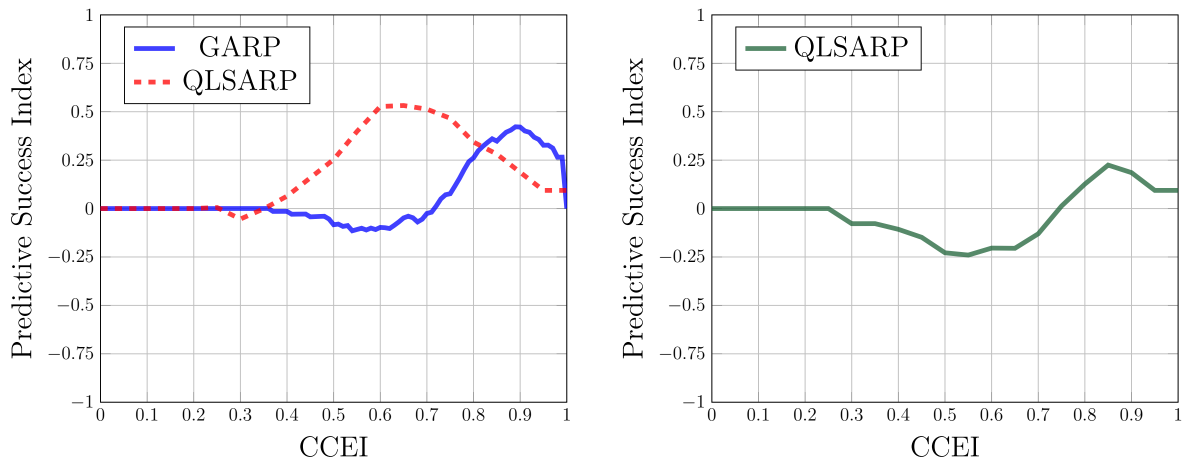
[16] The complete explanation of the experimental design and procedures is in the Appendix.

[17] We firstly select a random order of five goods, then generate a share of income spent on the first of them using uniform random distribution. The share of the remaining income is then determined in equal manner for the second good. The same procedure is repeated for the third, fourth and fifth good.

[18] Distributions are different according to Kolmogorov-Smirnov test with $p < .001$. The difference in means is significant with $p < .001$ according to t-test. Median GARP CCEI for real choices is .85 and .76 for random choices, the difference is significant according to Wilcoxon rank sum test with $p < .001$.

[19] Distributions are different according to Kolmogorov-Smirnov test with $p < .001$. The difference in means is significant with $p < .001$ according to t-test. Median QLSARP CCEI for real choices is .73 and .49 for random choices, the difference is significant according to Wilcoxon rank sum test with $p < .001$.

[20] Results are available from the authors upon request.



(a) H_0: Random Choices

(b) H_0: Random Choices that Pass GARP

Figure 10: Predictive Success Index

shows the predictive success index for QLSARP conditioning on random choices passing GARP. We observe that the predictive success index for QLSARP peaks for CCEIs levels between $.6 - .7$. This implies that to accept the hypothesis that subjects behave as if they had quasi-linear preference we would need to accept that they are willing to waste about 30 to 40% of their income. This is equivalent to losing an average of \$16 out the average experimental payments of \$40. Calculating the predictive success on random choices that pass GARP reduces the values of predictive success index. At the same time the predictive success index peaks at a CCEI level of .85. Our experimental evidence suggests that quasi-linearity of preferences in money is a strong assumption that deserves further investigation.

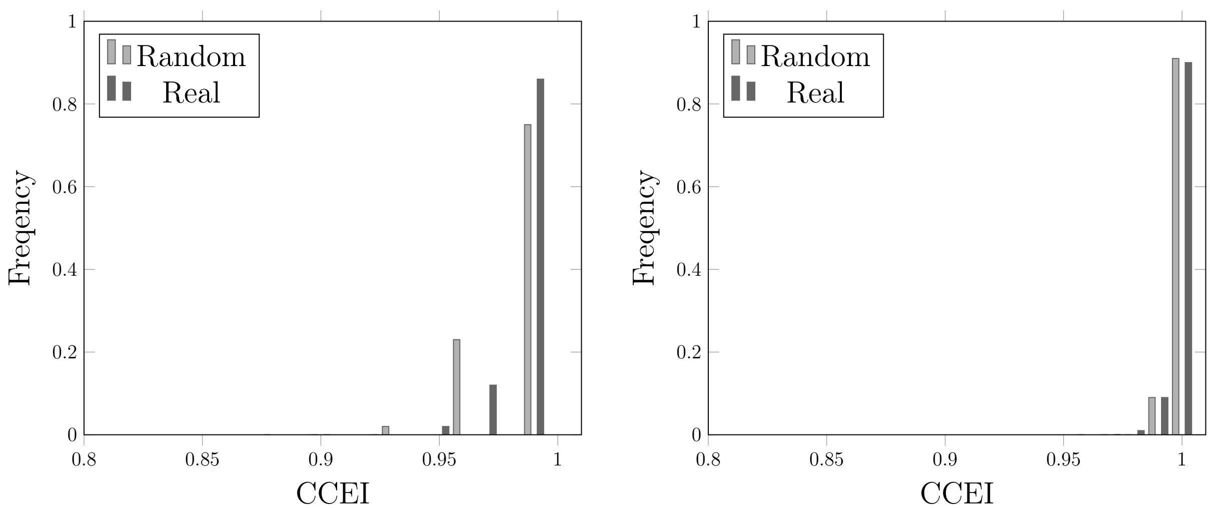
3.3 Quasi-Linearity in Panel Data

We use data from the Spanish Continuous Family Expenditure Survey (the Encuesta Continua de Presupuestos Familiares - ECPF). ECPF is a quarterly survey of households that are randomly rotated out at a rate of 12.5% per quarter. In this panel, household can be followed for up to eight consecutive periods and the years we use run from 1985 to 1997. For comparability with previous studies, we use the subsample used by Beatty and Crawford (2011) which comprises only two-adult households with a single income earner in the non-agricultural sector. The dataset consists of 21,866 observations on 3,134 households which gives an average of seven consecutive periods per household. Expenditures of each houselds are aggregated into five groups: "Food, Alcohol and Tobacco","Energy and Services at Home","Non Durables","Travel" and "Personal Services". The price data are national consumer price indexes for the corresponding expenditure categories.

Note that the longitudinal nature of the data requires assuming that preferences are stable over time. Also, the data set imputes the same price index for all the household for a given quarter. Price variation across households is then obtained through their rotation in the sample.

A common problem in testing rationality with household expenditure data is the low power of the test due to limited price variation. The power of the test can be increasing by imposing additional assumptions like quasi-linearity of preferences. For instance, Heufer et al. (2014) shows that the hypothesis of homethetic preferences is well-powered in the ECPF panel. The same is true for the test of quasi-linear preferences.

We first determine the power of our test of quasi-linear preferences by obtaining CCEIs for agents choosing at random. We generate random choices over the budgets using the same procedure as in our experiment (see previous section).

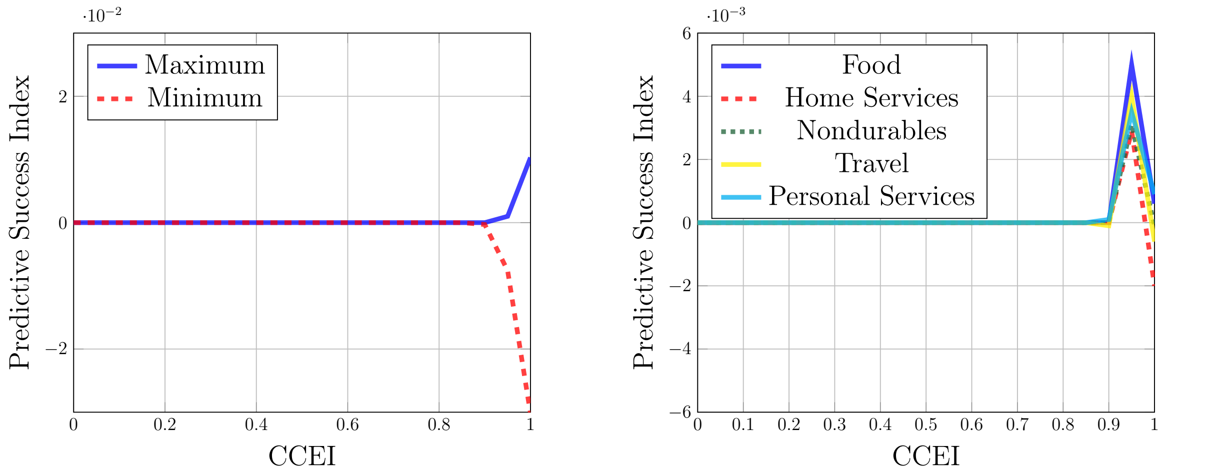


(a) Minimum CCEI

(b) Maximum CCEI

Figure 11: CCEI Distributions for QLSARP

Figure 11 shows the distributions of CCEIs for random and actual decisions, since there are five commodities and quasi-linearity of preferences could hold for each one of them, we report the distribution of maximum and minimum CCEI. For each agent we compute the CCEI for each commodity and take the maximum and minimum of them. These are the numbers used in Figure 11. This amounts to assuming that different agents might have quasi-linear preferences in different goods. Figure 11(a) shows the distributions of the maximum CCEIs for observed and random decisions. The mean of the maximum CCEIs for random decisions is .995 (the median is .997) and the mean of the maximum CCEIs for observed choices is .997 (the median is .997). Figure 11(b) shows the distributions of the minimum CCEIs for observed and random choices. The mean of the minimum CCEIs for random decisions is .979 (the median is .981) and the mean of the minimum CCEIs for random choices is .988 (the median is .989). If we consider CCEIs by commodity groups for random subjects, the mean is .988 (the median is .991) and for observed decisions the mean CCEI by the commodity groups is .992 (the median is .995).



(a) Bound for Predictive Succes Index (b) Predictive Sucess Indexes for Commodity Groups

Figure 12: Predictive Success Index

Figure 12 shows the predictive success indexes. Figure 12(a) shows the minimum and maximum predictive success indexes conditioned on random choices. Note that due to the very small difference between the CCEI for random and real subjects the predictive success is zero for the most levels of the CCEI. A difference appears for a CCEI level close to one. At this level, observed decisions are consistent with QLSARP while random decisions are not. The predictive success index peaks at the CCEI level of 1 and provide weak positive evidence for quasi-linearity (the index is .1). Figure 12(b) shows the predictive success index conditioned on random choices by commodity groups. In this case, the predictive success is close zero at every level of CCEI. The maximum among these predictive success indexes peaks at .005 at the CCEI level of .95. Not surpringly, the hypothesis of quasi-linear preferences does no worse once we assume that different people have quasi-linear preferences in different goods.

Quasi-linearity in money is not testable in the ECPF dataset since no money leftover is recorded. However, quasi-linearity of preferences are testable if we allow for the existence of an unobserved commodity Cherchye et al. (2015). We test for the joint hypothesis of quasi-linearity and the existence of an unobserved commodity and find that only 0.9% of households passed the test.[21] This suggests that our original findings are robust.

[21] We cannot calculate how severe deviations from rationality are because income is not observed.

4 Nonparametric Welfare Analysis

Afriat (1977) shows that revealed preference restrictions can be used to estimate the welfare effect of price changes. This idea was further developed by Varian (1982) to construct "support sets", i.e. the sets that include all the potential choices of consumer if the consumer is rational. We conduct a similar exercise to show how the assumption of quasi-linearity can help narrow prediction of choices in new budgets.

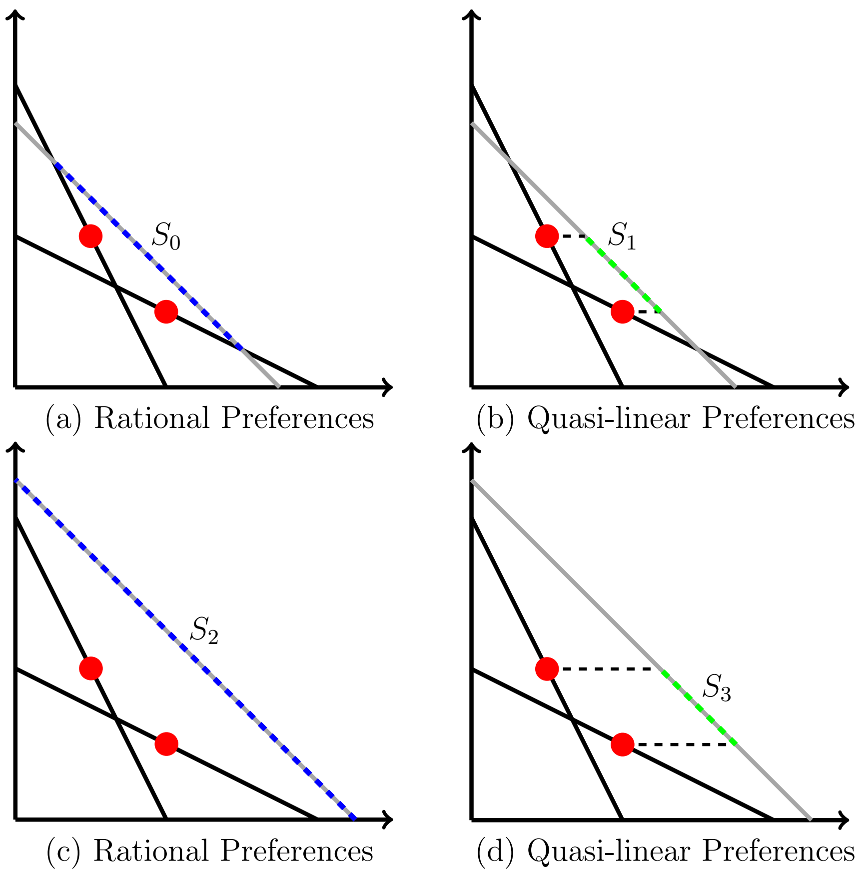
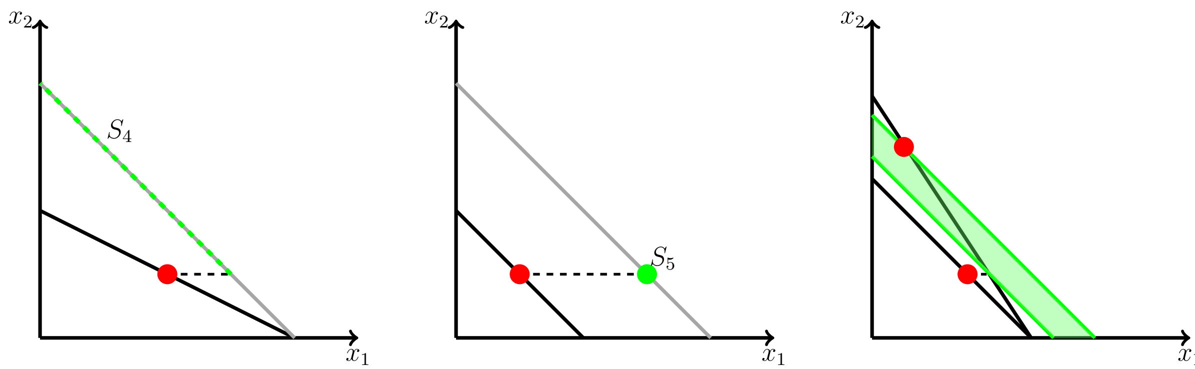


Figure 13: Possible Choices: Comparison of Assumptions of Rational Preferences and Quasi-Linear Preferences

Figure 13 shows the extrapolation of behavior based on GARP alone. The red dots are the chosen bundles and the dashed areas, denoted by S_i, are the sets of possible choices over the new budgets. Figure 13(a) shows all possible choices if preferences are rational (S_0). Any bundle not in S_0 violates GARP. Figure 13(b) shows all possible choices if preferences are quasi-linear (S_1). Note that all the points in $S_0 \setminus S_1$ would violate QLSARP, but not GARP. Figure 13(c) shows all possible choices if preferences are rational (S_2). In this case S_2 is the entire budget line, since GARP imposes no restriction choices on budgets that do not intersect with previous ones. Figure 13(d) all possible choices if preferences are quasi-linear

(S_3). Note that predictions under quasi-linearity remain the same regardless the new budget intersects with and old one or not.



(a) Wide Forecast (b) Narrow Forecast (c) Compensated Variation

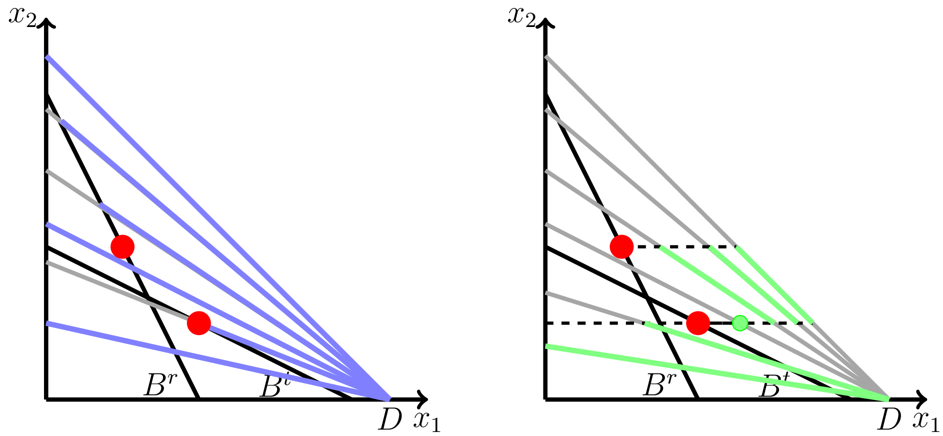
Figure 14: Possible Choices: Constructing a Forecast from a Single Observation

Figure 14 shows that if preferences are quasi-linear even one observed choice can help narrow the set of possible choices in new choice sets. Figure 14(a) present the case of a change of the price for one of the goods. In this case, amount of good x_2 have to be "above" the amount consumed in the original choice set (S_4). Figure 14(b) is the case of a change income holding prices constant. In this case, the set of possible choices is a singleton (S_5).[22] Sharper predictions can be obtained if additional choices are observed. Figure 14(c) shows that predictions can be narrowed further in this case. Figure 14(c) shows the bounds for *compensated variation* for the point in the outer budget after an increase in the price of good 2. The outer green line represent the budget that attains the original choice at the new prices. The utility associated with this budget cannot be lower that the utility at the original prices and therefore represent an upper bound of the compensation necessary to maintain the original level of utility. To derive the lower bound of the compensating variation consider the case in which all the points on the original budget set that give at least as much of good 2 as the observed choice in the new budget are indifferent to the originally chosen bundle. This certainly does not contradict quasi-linearity. In this case, the needed compensation is defined by the lower green line.[23]

The next step is to construct bounds on demand functions and consumer surplus. Figure 15 provides a two-dimensional example. Since under quasi-linearity in x_1, estimates of consumer surplus can be approximated by changes in the demand of x_2 due to price changes, we construct bounds for the demand in x_2.

[22]Note that if we relax the requirements to QLGARP, then the support set in this case would include the entire new budget set.

[23]This derivation is based on Figure 14(b), therefore, it would not be correct under the assumption of QLGARP instead of QLSARP.



(a) for GARP (b) for QLSARP

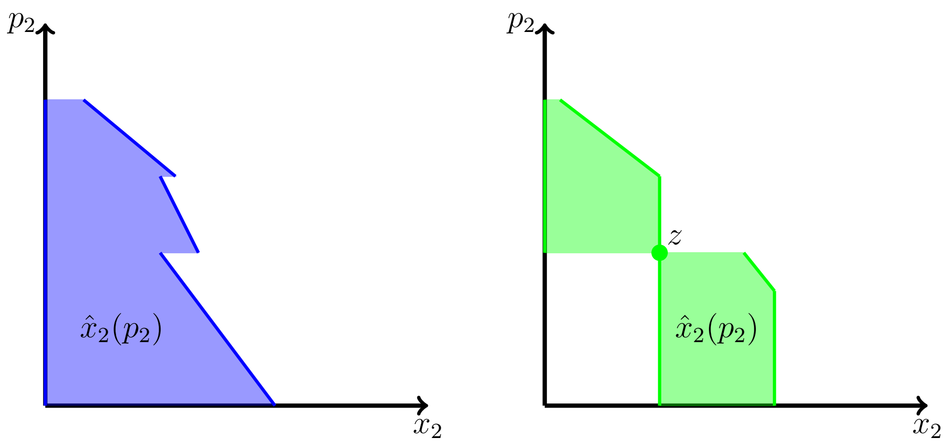
Figure 15: Constructing Demand Bounds

Figure 15 shows the procedure to bound the demand of x_2 according to GARP and QLSARP. Observed choices are represented by red dots the budgets B^t and B^r. Hypothetical budgets are represented by blue lines starting at point D. The only difference among these budgets is the price of x_2. Figure 15(a) shows bounds on the demand of x_2 in the case of rational preferences. Note that the bounds on demand will have two “jumps”. One at the intersection of the new budget sets with B^t and one at the intersection of the new budget sets with B^r. Note that these extrapolation is done over a set of non previously observed choice sets.

Figure 15(b) repeats the same exercise under the assumption of quasi-linear preferences. There are three different cases. The first case corresponds to the budget sets that intersect the actual choice in B^t. In this case the bound on demand do not depend on prices. The second case corresponds to the case in which the new budget set is parallel to B^r. This case is similar to Figure 14(b) where a point-prediction is possible. The third case corresponds to the new budget sets for which the choice at B^r is not affordable. The prediction in this case is bounded above by the consumption of x_2 in B^r and strictly below this amount when this is not affordable.

Figure 16 shows the bounds for the demand function constructed using the same procedure as the one used to construct Figure 15. Note that demand is not necessarily continuous. Two sectors on the bounds on demand deserve attention. Figure 16(a), which shows the bounds on demand based on rational preferences only, have jumps corresponding to the cases where choices have jumps as well.

Figure 16(b) shows bound for the demand correspondence given the assumption of quasi-linear preferences. In this case, there is a point at which bounds on demand shift to the left discontinuously. If the price is higher than the price at this point (z), then the lower bound



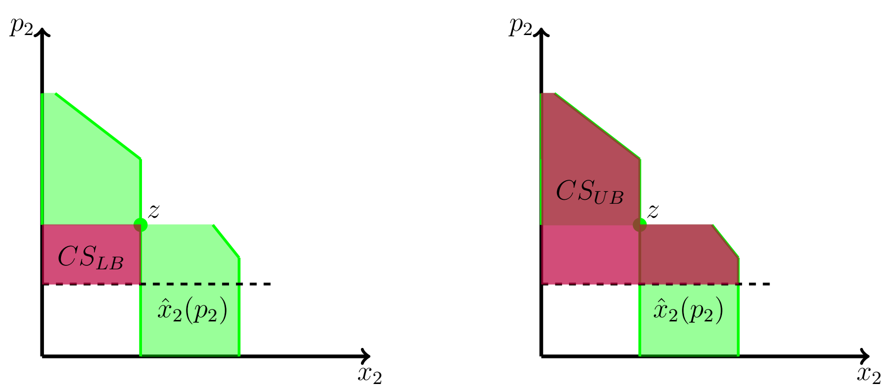
(a) Rational Preferences (b) Quasi-linear Preferences

Figure 16: Bounded Demands

for the demand is zero. If the price is lower than the price at z, then the lower bound for the demand is constant.

This exercise shows that the bounds under quasi-linearity can be tight than under the assumption of rational preferences alone.

The bounds on the demand function imply also bounds on consumer surplus. This can be done by using the upper bound of demand to construct the upper bound of the consumer surplus and the lower bound of demand to construct the lower bound of the consumer surplus. Note that upper bounds for demand functions are the right border lines of the green area. This boundary is weakly decreasing and therefore a valid demand function under quasi-linearity. This therefore provides the logical upper bound to all demand functions consistent with quasi-linearity and the original choices.



(a) Lower Bound for Consumer Surplus (b) Upper Bound for Consumer Surplus

Figure 17: Constructing Bounds for Compensated Variation and Consumer Surplus

Figure 17 shows the bounds for Consumer Surplus when preferences are quasi-linear.

Figure 17(a) shows the lower bound for the consumer surplus and Figure 17(b) shows the upper bound for the consumer surplus. The bounds on the consumer surplus follow tightly the bounds on the demand functions explained above.

5 Connection to Previous Literature

In this section we show connection between QLSARP and the test from Brown and Calsamiglia (2007). We show that if budgets are linear, then QLSARP is equivalent to the condition from Brown and Calsamiglia (2007). Therefore, if a consumption experiment satisfies QLSARP, then there is a strictly concave, continuous, strictly monotone and quasi-linear utility function that rationalizes the consumption experiment. This implies that if budgets are linear, then continuity and concavity have no empirical content under the assumption of quasi-linearity of preferences. Moreover, it allows us to to develop the test for concavity of preferences under the assumption of quasi-linearity preferences, using non-linear budgets.

Brown and Calsamiglia (2007) use linear budgets and normalize a price of a good in which preferences are supposed to be quasi-linear (numeraire), so the linear consumption experiment can be defined as $((x^t, y^t), (p^t, 1))_{t=1}^T$, where $p^t \in \mathbb{R}_+^{N-1}$ is the vector of prices. Linear budget set can be defined as $B^t = \{x \in \mathbb{R}^{N-1}, y \in \mathbb{R} :\ (p^t, 1)(x, y) \leq (p^t, 1)(x^t, y^t)\}$. Following Brown and Calsamiglia (2007) a linear consumption experiment $(x^t, p^t)_{t=1}^T$ is said to be **strictly cyclically monotone** if for any subset $\{((x^m, y^m), (p^m, 1))\}_{m=1}^M$ $p^1(x^2 - x^1) + p^2(x^3 - x^2) + \ldots + p^M(x^1 - x^M) > 0$. A utility function **rationalizes** the consumption experiment $((x^t y^t), (p^t, 1))_{t=1}^T$ if $u(x^t, y^t) \geq u(x, y)$ for any $p^t x + y \leq p^t x^t + y^t$ for any $t = 1, \ldots, T$.

Theorem 2 (Theorem 2.2 from Brown and Calsamiglia (2007))**.** *A finite consumption experiment $((x^t, y^t), (p^t, 1))_{t=1}^T$ can be rationalized by a strictly concave, continuous, quasi-linear and strictly monotone utility function if and only if it is strictly cyclically monotone.*

Lemma 1. *Let $((x^t, y^t), (p^t, 1))_{t=1}^T$ be a finite consumption experiment that generates a revealed preference relation R_v. Then, an acyclic R_v satisfies QLSARP if and only if the consumption experiment is strict cyclically monotone.*

Note that if there is a strictly concave, continuous, quasi-linear and strictly monotone utility function that rationalizes the consumption experiment, then there is a complete extension of the corresponding revealed preference relation that is complete, transitive and monotone and quasi-linear. Therefore, if the consumption experiment is strict cyclically monotone, then the revealed preference relation generated by the consumption experiment satisfies QLSARP. Therefore, we left to prove the reverse, i.e. if there is an acyclic R_v satisfies QLSARP, then the consumption experiment is strict cyclically monotone.

Proof. As in cyclical monotonicity for QLSARP we also need to consider any subset of budget experiment $\{((x^m, y^m), (p^m, 1))\}_{m=1}^M$ if for every $j < M$ $((x^j, y^j), (x^{j+1}, y^{j+1} - \beta^j) \in T(R_v)$, then $((x^M, y^M), (x^1, y^1 + \sum_{j=1}^{M-1} \beta^j)) \notin P(T(R_v))$. QLSARP implies this it should be true for any sequence[24] of β^j. However, checking the minimum β^j is enough. And this β^j can be defined as $\beta^j = p^j(x^{j+1} - x^j) + (y^{j+1} - y^j)$. Then $((x^M, y^M), (x^1, y^1 + \sum_{j=1}^{M-1} \beta^j)) \notin P(T(R_v))$ can be rewritten as $p^M x^1 + y^1 + \sum_{j=1}^{M-1} \beta^j > p^M x^M + y^M$. And simplifying the expression we get that $p^1(x^2 - x^1) + p^2(x^3 - x^2) + \ldots + p^M(x^1 - x^M) + \underbrace{(y^2 - y^1) + (y^3 - y^2) + \ldots + (y^1 - y^M)}_{=0} =$ $p^1(x^2 - x^1) + p^2(x^3 - x^2) + \ldots + p^M(x^1 - x^M) > 0$ that is stict cyclical monotonicity. □

Hence, combining Theorem 2 and Lemma 1 we can get the following result.

Theorem 3. *A finite consumption experiment $((x^t, y^t), (p^t, 1))_{t=1}^T$ can be rationalized by a strictly concave, continuous, quasi-linear and strictly monotone utility function if and only if R_v is acyclic and satisfies QLSARP with respect to n-th component (y^t).*

Note that Theorem 2 uses strict concavity of the utility function as a crucial assumption. While from Theorem 1 we know that QLSARP is equivalent to the existence of an extension of the revealed preference relation that is just complete, transitive, monotone and quasi-linear. Therefore, if budgets are linear concavity and continuity has no empirical content under the assumption of quasi-linearity of preferences.

Now let us show how to construct the test for the concavity of the utility function under the assumption of quasi-lienarity of preferences using non-linear budgets. Following Forges and Minelli (2009) a function $g^t(x, y) : \mathbb{R}^L \to \mathbb{R}$ is said to be **gauge function** of the budget set B^t if $B^t = \{x \in X : g^t(x, y) \leq 0\}$. And with a finite consumption experiment one can associate the linearized experiment, if gauge function is differentiable at least at the optimal points. $((x^t, y^t), C^t)_{t=1}^T$ is a **linearized consumption experiment** associated with $((x^t, y^t), B^t)_{t=1}^T$ if $C^t = \{x \in X : \nabla g^t(x^t, y^t)(x, y) \leq \nabla g^t(x^t, y^t)(x^t, y^t)\}$. Denote by R_v^C the revealed preference relation generated by the linearized consumption experiment $((x^t, y^t), C^t)_{t=1}^T$.

Theorem 4. *Let $((x^t, y^t), C^t)_{t=1}^T$ be a finite consumption experiment with gauge functions that are increasing, continuous, quasi-convex, differentiable at every (x^t, y^t) and can be represented as $g^t(x, y) = h^t(x) + y$. Then there is a continuous, strictly concave, strictly increasing and quasi-linear utility function ($u(x) + y$) that rationalizes it if and only if R_v^C is acyclic and satisfies QLSARP.*

Since the proof is very similar to the proof of the Theorem 2 let us sketch it to emphasize the main differences. The proof can be done just by showing that the following are equivalent:

[24] QLSARP specifies β^j to be positive, however, in Appendix A we show that to obtain quasi-linear representation, the similar condition should hold for any real sequence of β^j not necessarily positive.

(1) There is a continuous, concave, increasing and quasi-linear utility function that rationalizes R_v,

(2) There are numbers u^t and u^s such that $u^s \leq u^t + \nabla h^t(x^t)(x^s - x^t)$ for any $t, s = 1, \ldots, T$,[25]

(3) Linearized experiment satisfies cyclical monotonicity

(1) $\Rightarrow$ (2) From the first order condition for quasi-linear utility function we know that $\nabla u(x^t) = \nabla h^t(x^t)$ and from the fact that u is concave we can conclude that $\nabla h^t(x^t)$ is the super-gradient, i.e. $u(x) \leq u(x^t) + \nabla h^t(x^t)(x^s - x^t)$ for any $x \in X$.

The rest of the implications are the same as in the original proof, since we take $\nabla h^t(x^t)$ as linear prices, and as the result we will get the utility function that represents R_v^C and, hence, represents R_v. As we have already shown if prices are linear, then QLSARP is equivalent to strict cyclical monotonicity. That completes the proof. Hence, concavity of quasi-linear representation can be rejected if the consumption experiment satisfies QLSARP, while its linearized version does not. It would imply that there is a quasi-linear extension of the revealed preference relation, while there is no concave and quasi-linear utility function that rationalizes the consumption experiment.

6 Summary

We provide a necessary and sufficient condition for a set of observed choices to be consistent with the existence of complete, transitive, monotone and quasi-linear preference ordering. This condition applies to choices over compact and downward closed budget sets and do not require for preferences to be convex. This condition does not guarantee existence of a utility function but only the existence of a preference relation, that is complete transitive, monotone, quasi-linear and generates the observed behavior. This condition is sufficient for the existence of quasi-linear utility function if budget sets are linear.

We conduct a laboratory experiment to test the hypothesis that preferences are quasi-linear in money. Our experiments show that while individual choices are generally consistent with GARP they are no more consistent with quasi-linearity in money than choices made at random. We also use the data from the Spanish Continuous Family Expenditure Survey to test for quasi-linearity using household consumption data. We find support for the hypothesis of quasi-linearity against the the alternative that choices are made at random. However, and

[25] For the case of not necessarily differentiable utility function one similarly can start from $u^s + y^s \leq u^t + y^t + (\nabla h^t(x^t), 1)((x^s, y^s) - (x^t, y^t) = u^t + y^t \nabla h^t(x^t)(x^s - x^t) + (y^s - y^t) = u^t + y^s + \nabla h^t(x^t)(x^s - x^t)$, that is equivalent to the original inequality.

due to low observe variation in prices, we find no support for the existence of quasi-linear preferences against the alternative of the existence of a smooth utility function.

We discuss how the assumption of quasi-linearity can be used to perform non-parametric welfare analysis. In particular, we show how to estimate the bounds for the demand correspondence, consumer surplus and compensated variation under the assumption of quasi-linear preferences. Note that this estimations requires only the existence of a complete, transitive, monotone and quasi-linear preference ordering. These bounds can be considerably tighter than nonparametric bounds derived under the assumption of rational preferences alone.

An important advantage of the test of quasi-linearity we present here is that it applies to a large class of problems. First, it applies to consumer problems in the presence of distortions due to taxes, subsidies or non-linear pricing. Second, it can also be extended to test for quasi-linearity of preferences in strategic situations where such assumption might be invoked (e.g. auction theory).

Appendix

Appendix A: Proof of Theorem 1

Recall the statement of Theorem 1.

Theorem 1. *An acyclic revealed preference relation R_v generated by a finite consumption experiment with monotone and compact budgets has an extension that is complete, transitive, monotone and quasilinear in the i-th component if and only if R_v satisfies QLSARP with respect to the i-th component.*

To prove Theorem 1 we use the notion of function over preference relations. A convenient example of such function is the *transitive closure*, which adds (x,z) to R for each $(x,y) \in R$ and $(y,z) \in R$. Being more precise $(x,y) \in T(R)$ if there is a sequence of elements $S = s_1, \dots, s_n$, such that for every $j = 1, \dots, n-1$ $(s_j, s_{j+1}) \in R$, where T stays for transitive closure. That is transitive closure of a preference relation is a transitive preference relation, since transitive closure is idempotent and applying transitive closure to the transitive closure of the relation does not add anything to the relation. The function we propose is a generalization of transitive closure that guarantees that every fixed point of it is transitive, monotone and quasi-linear preference relation.

The proof is organized as follows. First, we introduce the terminology and define the function over preference relations that implies the desirable properties while extending the preference relation. Further we show that the function is "well-behaved", i.e. the possibility of extension of preference relation by the function can be stated as a simple set-theoretical condition. Finally, we show that consistency of a preference relation with the function is equivalent to QLSARP.

Let $F : \mathcal{R} \to \mathcal{R}$ be a function over preference relations. For a given function $F(R)$, a preference relation R is said to be F-**consistent** if $F(R) \cap P^{-1}(R) = \emptyset$.[26]

Lemma 2. *A preference relation $R \subseteq F(R)$ is F-consistent if and only if $R \preceq F(R)$.*[27]

We omit the proof since it can be found in Demuynck (2009). Henceforth, further we use these notions as equivalent definitions. For any function $F : \mathcal{R} \to \mathcal{R}$, let $\mathcal{R}^*_F = \{R \in \mathcal{R} | R \preceq F(R)\}$. We use the following definition throughout the section.

Definition 3. *A function $F : \mathcal{R} \to \mathcal{R}$ is said to be an* ***algebraic closure*** *if*

(1) For all $R, R' \in \mathcal{R}$, if $R \subseteq R'$, then $F(R) \subseteq F(R')$, and,

(2) For all $R \in \mathcal{R}$, $R \subseteq F(R)$, and,

[26]Recall that $(x,y) \in P^{-1}(R)$ if $(y,x) \in R$ and $(y,x) \notin R$.

[27]Recall that $R \preceq F(R)$ ($F(R)$ is an extension of R) if $R \subseteq F(R)$ and $P(R) \subseteq P(F(R))$.

(3) For all $R \in \mathcal{R}$, $F(F(R)) \subseteq F(R)$, and,

(4) For all $R \in \mathcal{R}$ and all $(x, y) \in F(R)$, there is a finite relation $R' \subseteq R$, s.t. $(x, y) \in F(R')$.

Properties (1) to (3) are those that define **closure** and are connected to the extrapolation of the relation by $F(R)$. Property (1) is *monotonicity*, it states that from larger amount of information we can get better extrapolation. Property (2) is *extensiveness*, that is function adds additional information about preference relation. Property (3) is *idempotence*, that is the function delivers all the information after the first application of it to the binary relation. Property (4) is one that makes the closure **algebraic** (for the formal definition in more general context see e.g. Crawley and Dilworth (1973)). This property is one that allows us to make our theory testable with finite data set, i.e. there is finite set of observations that is not F-consistent.

A function $F : \mathcal{R} \to \mathcal{R}$ is said to be **weakly expansive**[28] if for any $R = F(R)$ and $N(R) \neq \emptyset$, there is $T \subseteq N(R)$ such that $R \cup T \in \mathcal{R}^*_F$. This property states that F is non-satiated, i.e. for any incomplete fixed point ($F(R) = R$) preference relation, there is F-consistent extension of this relation. It is important since we are used to assume that preferences are complete, i.e. any two bundles are comparable, and weak expansiveness guarantees that the set of assumptions is not contradictory with completeness axiom.

Theorem 5 (Theorem 2 from Demuynck (2009))**.** *Let F be a weakly expansive algebraic closure. Then, a relation $R \in \mathcal{R}$ has a complete extension $R^* = F(R^*)$ if and only if R is F-consistent.*

To provide the intuition for the proof of Theorem 5 let us show the algorithm that can is used to construct a fixed point complete extension. Denote $R_0 = R$, i.e. the relation we start from. Then, for any $a > 0$ if $R_a \neq F(R_a)$, then $R_{a+1} = F(R_a)$. If $R_a = F(R_a)$, then from expansiveness we know that there is T, such that $R_a \cup T \in \mathcal{R}^*_F$, so let $R_{a+1} = R_a \cup T$. The existence of the limit relation which is a fixed point of F is guaranteed by the fact that F is algebraic closure.

Let us specify the closure that will guarantee us existence of complete, monotone, transitive and quasi-linear in i-th component extension of the original relation.

Definition 4. *Denote the* ***Quasi-linear in the i-th component Monotone Transitive closure*** *by $QLiMT(R)$. Then, $(x, y) \in QLiMT(R)$ if there is a sequence $S = s_1, ..., s_n$, s.t. $s_1 = x$, $s_n = y$ and $\forall j = 1..n-1$*

$$\exists \alpha_j \in \mathbb{R} : (s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R, \text{ or}$$

[28] Weak expansiveness is equivalent to the condition C7 in Demuynck (2009).

$s_j \gg s_{j+1}$

We consider quasi-linearity[29] with monotonicity, since these notions are usually considered together (see e.g. Kreps (2012) who defines quasi-linearity as the definition we provide jointly with the monotonicity). However, monotonicity has no empirical content if we assume budgets to be monotone. In the case of non-monotone budgets QLSARP can be easily modified by using $TM(R)$ instead of $T(R)$.[30] $TM(R)$ is transitive and monotone closure, that can be defined as following: $(x, y) \in TM(R)$ if there is a sequence $S = s_1, \ldots, s_n$, such that for any $j = 1, \ldots, n-1$ either $(s_j, s_{j+1}) \in R$ or $s_j \geq s_{j+1}$. If one wants to separate the assumption of quasi-linearity from the monotonicity, monotonicity of budget sets can be relaxed to monotonicity in the i-th component[31] (the same as it should be quasi-linear in). In this case QLSARP is equivalent to the existence of the extension of revealed preference relation that is quasi-linear in the i-th component, monotone in the i-th component, transitive and complete .

The proof of Theorem 1 consists of two parts:

(i) Proposition 1 shows that $QLiMT(R)$ is a weakly expansive algebraic closure;

(ii) Proposition 2 shows that R_v satisfies QLSARP if and only if $T(R_v)$[32] is $QLiMT$-consistent.

After proving these Propositions we can apply Theorem 5 to complete the proof of Theorem 1.

(i) Proof of Proposition 1

Proposition 1. *$QLiMT(R)$ is a weakly expansive algebraic closure.*

The proof of the Proposition 1 consists of the following lemmas:

(1) Lemma 3 shows that $R = QLiMT(R)$ if and only if R is quasi-linear, transitive and monotone. This Lemma shows that any fixed point of $QLiMT$ is a quasi-linear, transitive and monotone relation. Moreover, we use it extensively in the further proofs.

(2) Lemma 4 shows that $QLiMT$ is a closure, i.e. satisfies (1)-(3). Note that if it is a closure, it is algebraic by definition, because any element of $QLiMT(R)$ is added by adding the finite sequence of elements of R which generates it.

[29]Recall that R is said to be **quasi-linear in the i-the component** if for any $(x, y) \in R$ and $\alpha \in \mathbb{R}$ $(x + \alpha e_i, y + \alpha e_i) \in R$.

[30]Recall that $(x, y) \in T(R)$ if there is a sequence $S = s_1, \ldots, s_n$, $s_1 = x$ and $s_n = y$, such that for any $j = 1, \ldots, n-1$ $(s_j, s_{j+1}) \in R$

[31]The budget is monotone with respect to i-component if $x \in B$ implies that for any $\alpha \in \mathbb{R}_+$ $x - \alpha e_i \in B$

[32]Recall that R_v is a revealed preference relation obtained $(x^t, y) \in R_v$ if $y \in B^t$, where x^t is chosen bundle and B^t is corresponding budget. $T(R_v)$ is a transitive closure of revealed preference relation.

(3) Lemma 5 shows that $QLiMT$ is weakly expansive

Lemma 3. *$R = QLiMT(R)$ if and only if R is quasi-linear, transitive and monotone.*

Proof. ($\Rightarrow$). R is transitive, because if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in QLiMT(R)$. R is monotone also by the definition of $QLiMT(R)$. R is quasi-linear in i-th component, i.e. if $(x + \alpha e_i, y + \alpha e_i) \in R$, then $(x, y) \in QLiMT(R)$.

($\Leftarrow$). From the Definition of $QLiMT(R)$ it is clear that $R \subseteq QLiMT(R)$. Therefore, we need to show that $QLiMT(R) \subseteq R$ to show the equality of these sets. To prove this we need to show, that if there is $(x, y) \in QLiMT(R)$, then $(x, y) \in R$. To do this take $(x, y) \in QLiMT(R)$ and show that (x, y) is in R as well. We prove this by the induction on the length of chain that adds (x, y) to $QLiMT$. If length of the shortest chain is 2 it is immediately true since R is quasi-linear, transitive and monotone. Now suppose that every element $(x, y) \in QLiMT(R)$, such that it can be added to the $QLiMT(R)$ by a chain of the length no more than k is in R as well. To do the induction step, consider an element $(x, y) \in QLiMT(R)$ that is added to the $QLiMT(R)$ by a chain of the length of $k + 1$. Let us show that length of the chain can be reduced, i.e. the same element can be added to the $QLiMT(R)$ by a chain of the length no more than k. Take a $j \in \{1, \dots, n-1\}$ and consider the four following cases:

Case 1: $s_{j-1} \gg s_j$ and $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R$. Knowing that $s_{j-1} + \alpha_j e_i \gg s_j + \alpha_j e_i$ then by monotonicity, $(s_{j-1} + \alpha_j e_i, s_j + \alpha_j e_i) \in R$. By transitivity $(s_{j-1} + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R$. Therefore, length of the chain can be reduced.

Case 2: $s_{j-1} \gg s_j$ and $s_j \gg s_{j+1}$, then $s_{j-1} \gg s_{j+1}$. By monotonicity $(s_{j-1}, s_{j+1}) \in R$. Therefore, length of the chain can be reduced.

Case 3: $(s_{j-1} + \alpha_{j-1} e_i, s_j + \alpha_{j-1} e_i) \in R$ and $s_j \gg s_{j+1}$ Knowing that $s_j + \alpha_{j-1} e_i \gg s_{j+1} + \alpha_{j-1} e_i$ then by monotonicity, $(s_j + \alpha_{j-1} e_i, s_{j+1} + \alpha_{j-1} e_i) \in R$. By transitivity $(s_{j-1} + \alpha_{j-1} e_i, s_{j+1} + \alpha_{j-1} e_i) \in R$. Therefore, length of the chain can be reduced.

Case 4: $(s_{j-1} + \alpha_{j-1} e_i, s_j + \alpha_{j-1} e_i) \in R$ and $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R$. By quasi-linearity $(s_j + \alpha_{j-1} e_i, s_{j+1} + \alpha_{j-1} e_i) \in R$. By transitivity $(s_{j-1} + \alpha_{j-1} e_i, s_{j+1} + \alpha_{j-1} e_i) \in R$. Therefore, length of the chain can be reduced.

This completes the induction. □

Lemma 4. *$QLiMT(R)$ is an algebraic closure*

Note that $QLiMT(R)$ is algebraic (satisfies condition (4)) by construction, since every element can be added using a finite sequence. According to the Lemma 5 from Demuynck (2009) $F(R)$ is a closure if and only if $F(R) = \bigcap\{Q \supseteq R : \ Q = F(Q)\}$. Hence, it is enough to prove that $QLiMT(R) = \bigcap\{Q \supseteq R : \ Q = QLiMT(Q)\}$

Proof. (⊆) Let $(x, y) \in QLiMT(R)$, then there is a sequence $S = s_1, \ldots, s_n$ such that $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R$ or $s_j \gg s_{j+1}$ for any $j = 1, \ldots, n-1$. Since, $R \subseteq Q$, then for the entire sequence $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in Q$ or $s_j \gg s_{j+1}$ for any $j = 1, \ldots, n-1$. Hence, $(x, y) \in QLiMT(Q)$.

(⊇) Note that from Lemma 3 we know that $QLiMT(QLiMT(R)) = QLiMT(R)$. Therefore $QLiMT(R) \in \bigcap\{Q \supseteq R : \ Q = QLiMT(Q)\}$. Thus, if $(x, y) \in Q$ for any $Q \in \bigcap\{Q \supseteq R : \ Q = QLiMT(Q)\}$, then $(x, y) \in QLiMT(R)$. □

Lemma 5. *$QLiMT(R)$ is weakly expansive.*

Proof. Take a point $(x, y) \in N(R)$ and consider the relation $R' = R \cup \{(x, y)\}$. We need to show that $QLiMT(R') \cap P^{-1}(R') = \emptyset$. Assume to the contrary that there is a $(z, w) \in QLiMT(R') \cap P^{-1}(R') \neq \emptyset$. Note that from the assumptions we made $(x, y) \neq (z, w)$, since $(x, y) \in N(R)$ and $(w, z) \in P(R)$.

So, there is a chain $S = s_1, \ldots, s_n$ such that $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in R$ or $s_j \gg s_{j+1}$ for any $j = 1, \ldots, n-1$ and there is k such that $s_k + \alpha_k e_i = x$ and $s_{k+1} + \alpha_k e_i = y$ and $(w, z) \in R$ by assumption. So, let us consider the following sequence[33] $S' = s_{k+1}, \ldots, s_n, w, z, s_1, \ldots, s_k$ with:

$$(s_{k+1} + \alpha_{k+1} e_i, s_{k+2} + \alpha_{k+1} e_i), \ldots, (s_{n-1} + \alpha_{n-1} e_i, w + \alpha_{n-1} e_i),$$
$$(w, z), (z + \alpha_1 e_i, s_2 + \alpha_1 e_i), \ldots, (s_{k-1} + \alpha_{k-1} e_i, s_k + \alpha_{k-1} e_i)$$

elements of R.

From Lemma 3 we know that R is quasi-linear. Therefore, $(s_{k+1}, s_k) \in R$. Recall that by Lemma 3 R is quasi-linear. Hence, $(y, x) \in R$ since they can be obtained by adding $\alpha_k e_i$ to s_{k+1} and s_k respectively. Therefore, $(x, y) \notin N(R)$.

□

This completes the proof of Proposition 1.

(ii) Proof of Proposition 2: R_v satisfies QLSARP if and only if $QLiMT(T(R_v)) \cap P^{-1}(T(R_v)) = \emptyset$

Recall the definition of QLSARP.

Definition 5. *A revealed preference relation satisfies **QLSARP** with respect to i-th component if for any sequence of distinct elements $x^{k_1}, ..., x^{k_n} \in C$ and $(\alpha, \beta_3, \ldots, \beta_n) \in \mathbb{R} \times$*

[33] We omit elements of sequence that correspond to monotonicity, since $QLiMT(R) = R$ already implies that all monotonicity pairs are already in R.

$\mathbb{R}_{++} \times \ldots \times \mathbb{R}_{++}$, *such that* $(x^{k_1}, x^{k_2} - \alpha e_i) \in P(T(R_v))$ *and* $(x^{k_j}, x^{k_{j+1}} - \beta_{j+1} e_i) \in T(R_v)$ *for* $j = 2, \ldots, n-1$, *then* $(x^{k_n}, x^{k_1} + (\alpha + \sum_{j=3}^{n} \beta_j) e_i) \notin T(R_v)$.

And since $QLiMT(R)$ is a weakly expansive algebraic closure we need to show that QLSARP is equivalent to $QLiMT$-consistency for R_v. Recall that elements of $QLiMT$ are added through sequences, such that $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_j) \in T(R_v)$ or $s_j \gg s_{j+1}$. A sequence $S = s_1, \ldots, s_n$, $s_1 = x$ and $s_n = y$ that adds (x, y) to $QLiMT(R)$ is said to be (x, y)**-irreducible length sequence** if there is no shorter sequence $S' = s'_1, \ldots, s'_n$, $s'_1 = x$ and $s'_n = y$ that adds (x, y) to $QLiMT(R)$. For any sequence S denote by $\beta_j = \alpha_j - \alpha_{j-1}$, $j = 2, \ldots, n$. From the definition of $QLiMT(R)$ we can be sure that for any $(x, y) \in QLiMT(R)$ there is a finite irreducible length sequence. Note that if $(x, y) \in R$ the (x, y)-irreducible length sequence will trivially be $s_1 = x$, $s_2 = y$. Moreover, for each element there is the shortest sequence, hence we need to show that QLSARP checks that for any pair (x, y) from the strict part of $P(T(R_v))$ there is no (y, x)-irreducible length sequence in $T(R_v)$ - $(y, x) \in QLiMT(T(R_v))$.

But QLSARP does not say anything about one element being greater than another. So let us show that no (x, y)-irreducible length sequence will contain $s_j \gg s_{j+1}$. Note that assumption of monotonicity of budget sets allows us to claim that $(x, y) \in R_v$ then for any $z \leq y$ $(x, z) \in R_v$ as well. And this fact can be generalized for the transitive closure of the relation.

Fact 1. $(x, y) \in T(R_v)$ *then* $(x, z) \in T(R_v)$ *for any* $z \leq y$.

Hence, none of (x, y)-irreducible length sequences will contain $s_j \gg s_{j+1}$, because s_j has to be a chosen point, hence $(s_j, s_{j+1}) \in T(R_v)$ and the length of the sequence can be reduced otherwise.

For further proof denote by $\beta_j = \alpha_j - \alpha_{j-1}$ for $j \in \{2, \ldots, n\}$. Originally $QLiMT(R)$ assumes β_j to be any real number while QLSARP considers only positive β_j. So, let us show that (x, y)-irreducible length sequence will not contain $\beta_j \leq 0$.

Lemma 6. *For any* $(x, y) \notin T(R_v)$ *and* $(x, y) \in QLiMT(T(R_v))$, *an* (x, y)*-irreducible length sequence has all* $\beta_j > 0$.

Proof. On the contrary assume that there is $j \in \{2, ..., n-1\}$ such that $\beta_j \leq 0$. And further we show that there is a shorter sequence $S' = s'_1, \ldots, s'_{j-1}, s'_{j+1}, \ldots, s'_n$, $s'_1 = x$ and $s'_n = y$, such that for $k = 1, \ldots, j-1, j+1, \ldots, n'-1$, $(s'_k + \alpha'_k e_i, s'_{k+1} + \alpha'_k e_i) \in T(R_v)$. This contradicts the fact that the original sequence is (x, y)-irreducible length sequence.

If $\beta_j \leq 0$, then $\alpha_j \leq \alpha_{j-1}$, hence $s_j + \alpha_{j-1} e_i \geq s_j + \alpha_j e_i$. Therefore, $(s_{j-1} + \alpha_{j-1} e_i, s_j + \alpha_{j-1} e_i) \in T(R_v)$ (by construction), $(s_j + \alpha_{j-1} e_i, s_j + \alpha_j e_i) \in T(R_v)$ (by Fact 1, i.e. monotonicity of $T(R_v)$) and $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in T(R_v)$. Hence , by transitivity of $T(R_v)$ $(s_{j-1} + \alpha_{j-1} e_i, s_{j+1} + \alpha_j e_i) \in T(R_v)$.

So, we need to define s'_k and α'_k for all $k = 1, \ldots, j-1, j+1, \ldots, n$ to obtain $(s'_k + \alpha'_k e_i, s'_{k+1} + \alpha'_k e_i) \in T(R_v)$. Let $s'_k = s_k$ for every $k \neq j+1$ and $s_{j+1} = s_{j+1} - \beta_j$. Let $\alpha'_k = \alpha_k$ for every $k \notin \{j, j+1\}$. Then for every $k \neq j+1$ $(s'_k + \alpha'_k e_i, s'_{k+1} + \alpha'_k e_i) = (s_k + \alpha_k e_i, s_{k+1} + \alpha_k e_i) \in T(R_v)$. Let $\alpha'_{j+1} = \alpha_{j+1} + \beta_j \leq \alpha_{j+1}$, then (i) $(s_{j-1} + \alpha_{j-1} e_i, s'_{j+1} + \alpha_{j-1} e_i) \in T(R_v)$, and (ii) $s'_{j+1} + \alpha'_{j+1} e_i = s_{j+1} + \alpha_{j+1} e_i$. So, we only left to show that $(s_{j-1} + \alpha_{j-1} e_i, s'_{j+1} + \alpha_{j-1} e_i) \in T(R_v)$ to complete the proof. Since $\alpha'_{j+1} \leq \alpha_{j+1}$, then $s_{j+2} + \alpha'_{j+1} e_i \leq s_{j+2} + \alpha_{j+1} e_i$. Hence, by Fact 1 $(s_{j+1} + \alpha_{j+1} e_{i,j+2} + \alpha'_{j+1} e_i) \in T(R_v)$. Since $s'_{j+1} + \alpha'_{j+1} e_i = s_{j+1} + \alpha_{j+1} e_i$, then $(s_{j+1} + \alpha_{j+1} e_{i,j+2} + \alpha'_{j+1} e_i) = (s'_{j+1} + \alpha'_{j+1} e_{i,j+2} + \alpha'_{j+1} e_i) \in T(R_v)$.

□

Another difference between QLSARP and $QLiMT$ is that $QLiMT(R)$ does not require elements of sequence $S = s_1, \ldots, s_n$ to be distinct, while QLSARP implies there is no $s_j + \alpha_j e_i = s_k + \alpha_k e_i$.

Lemma 7. *For any $(x, y) \notin R_v$ and $(x, y) \in QLiMT(T(R_v))$ and (x, y)-irreducible length sequence S there is not $j \neq k$ such that $s_j + \alpha_j e_i = s_k + \alpha_k e_i$.*

Proof. Without loss of generality assume that $k > j$, then from Lemma 6 $\alpha_k > \alpha_j$. Hence $s_k \leq s_j$ and $(s_{j-1} + \alpha_{j-1} e_i, s_k + \alpha_{j-1} e_i) \in T(R_v)$. So, S is not (x, y)-irreducible length sequence. □

To prove Theorem 1 we need to show that R_v satisfies QLSARP with respect to i-th component if and only if $T(R_v)$ is $QLiMT$-consistent. Recall that $QLiMT$-consistency is equivalent to $QLiMT(T(R_v)) \cap P^{-1}(T(R_v)) = \emptyset$.

Proposition 2. *$QLiMT(T(R_v)) \cap P^{-1}(T(R_v)) = \emptyset$ in and only if R_v satisfies QLSARP with respect to i-th component.*

Proof. ($\Leftarrow$) We assume that there is a contradiction of $QLiMT$-consistency and construct the contradiction of QLSARP from the contradiction of $QLiMT$-consistency. Assume that $QLiMT(T(R_v)) \cap P^{-1}(T(R_v)) \neq \emptyset$, thus there is $(y, x) \in P(T(R_v))$ and $(x, y) \in QLiMT(T(R_v))$. Since $(x, y) \in QLiMT(T(R_v))$, there is (x, y)-irreducible length sequence $S = s_1, \ldots, s_n$, $s_1 = x$ and $s_n = y$, such that $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in T(R_v)$ and $s_1 = x$ and $s_n = y$. For $j = 1, \ldots, n$ let $x^{k_j} = s_j + \alpha_j e_i$ and for $j = 2, \ldots, n$ let $\beta_j = \alpha_j - \alpha_{j-1}$. Note that for $j = 1, \ldots, n-2$

(i) $x = x^{k_1} - \alpha_1 e_i$ implies $(y, x) = (y, x^{k_1} - \alpha_1 e_i) \in P(T(R_v))$ and $(x^{k_j}, x^{k_{j+1}} - \beta_{j+1} e_i) \in T(R_v)$;

(ii) $x^{k_j} \in C$ (all x^{k_j} are chosen points) by construction of R_v;

(iii) All $\beta_j > 0$, by Lemma 6;

(iv) All x^{k_j} are distinct points, by Lemma 7;

(v) $\alpha_j = \sum_{r=1}^{j} \beta_r + \alpha_1$ and $s_n = y$.

Then $(x^{k_n}, y + (\sum_{r=1}^{j} \beta_r + \alpha_1)e_i) \in T(R_v)$. That is exactly a contradiction of QLSARP.

($\Rightarrow$) Assume that there is a violation of QLSARP, let us show that then $T(R_v)$ is not $QLiMT$-consistent - $QLiMT(T(R_v)) \cap P^{-1}(T(R_v)) \neq \emptyset$. Let $y = x^{k_1}$ and $x = x^{k_2} - \alpha e_i$, then $(y, x) \in P(T(R_v))$. Let $\alpha_1 = \alpha$, $\alpha_j = \sum_{r=1}^{j} \beta_r + \alpha_1$ and $s_j = x^{k_j} - \alpha_j e_i$ for $j = 2, \ldots, n$. Then $x^{k_1} = x$, $x^{k_n} = y$ and for any $j = 2, \ldots, n$ there is $(s_j + \alpha_j e_i, s_{j+1} + \alpha_j e_i) \in T(R_v)$. Therefore, $(x, y) \in QLiMT(T(R_v))$. This implies that $QLiMT(T(R_v)) \cap P^{-1}(T(R_v)) \neq \emptyset$, i.e. there is a violation of $QLiMT$-consistency.

□

This allows us to apply Theorem 5 to complete the proof of Theorem 1.

Appendix B: An Extension Theorem for Weak Rationalization

Since Afriat (1967), in revealed preference theory it is common to talk about "weak rationalization" as a weakened concept of the rationalization we mentioned above. Weak rationalization assume that chosen point is only weakly preferred to the points that lie on the upper boundary of the budget set. However, it provokes a gap between the theoretical definition of revealed preference relation and the consumption experiment, because strict part can not be determined in standard way. Let us refine QLSARP to test for the existence of complete, quasi-linear, transitive and monotone extension of the weak rationalization.

For a monotone and compact set B, let ∂B be the **upper boundary** if for any $y \in \partial B$ and $z > y$, $z \notin B$. Denote the weak rationalization generated by finite consumption experiment $(x^t, B^t)_{t=1,\ldots,T}$ by R_w, then $(x^t, y) \in R_w$ if and only if $y \in B^t$ and $(x^t, y) \in P(R_w)$[34] if and only if $y \in B^t \setminus \partial B^t$. Let $T_w(R_w)$ be the **transitive closure of weak rationalization** and $(x, y) \in P(T_w(R_w))$ if and only if there is a sequence $S = s_1, \ldots, s_n$, $s_1 = x$ and $s_n = y$, such that for every $j = 1, \ldots, n-1$ $(s_j, s_{j+1}) \in R_w$ and there is k such that $(s_k, s_k + 1) \in P(R_w)$; $(x, y) \in I(T_w(R_w))$ if and only if there is a sequence $S = s_1, \ldots, s_n$, $s_1 = x$ and $s_n = y$, such that for every $j = 1, \ldots, n-1$ $(s_j, s_{j+1}) \in R_w$ and $(x, y) \notin P(T_w(R_w))$. Then $T_w(R_w)$ is consistent with definition of preference relation and can be decomposed into disjoint strict and indifference parts, such that $T_w(R_w) = P(T_w(R_w)) \cup I(T_w(R_w))$. This allows us to apply similar construction procedure to $T_w(R_w)$ as to $T(R_v)$.

Definition 6. *A weak rationalization R_w satisfies **QLGARP** with respect to i-th component if for any sequence of distinct elements $x^{k_1}, ..., x^{k_n} \in C$ and $(\alpha, \beta_3, \ldots, \beta_n) \in \mathbb{R} \times \mathbb{R}_{++} \times$*

[34] Recall that by $P(R)$ we denote strict part of the relation

$\ldots \times \mathbb{R}_{++}$, *such that* $(x^{k_1}, x^{k_2} - \alpha e_i) \in P(T_w(R_w))$ *and* $(x^{k_j}, x^{k_{j+1}} - \beta_{j+1} e_i) \in T_w(R_w)$ *for* $j = 2, \ldots, n-1$, *then* $(x^{k_n}, x^{k_1} + (\alpha + \sum_{j=3}^{n} \beta_j) e_i) \notin T_w(R_w)$.

A weak rationalization is said to be **weakly acyclic** if it satisfies GARP[35] and from Afriat (1967) we know that rationalization is weakly acyclic if and only if it satisfies GARP. Then, the extension result can be immediately achieved from Theorem 1

Corollary 1. *A weakly acyclic weak rationalization R_w generated by a finite consumption experiment with monotone and compact budgets has an extension that is complete, transitive, monotone and quasilinear in i-the component if and only if R_w satisfies QLGARP with respect to the i-th component.*

The proof of Corollary 1 is straight-forward, since the $T_w(R_w)$ is monotone and transitive as well as $T(R_v)$, then Lemma 6 and Lemma 7 hold.

Proposition 3. $QLiMT(T_w(R_w)) \cap P^{-1}(T(R_w)) = \emptyset$ *in and only if R_w satisfies QLGARP with respect to the i-th component.*

Then, Proposition 3 can be proven similarly to the Proposition 2. And the proof of Proposition 1 is done for an arbitrary preference relation. Then the proof of Corollary 1 follows immediately from Theorem 5.

Appendix C: Algorithmic Implementation of QLSARP

Let $a_{ij} \in \mathbb{R}$ be a minimum[36] number s.t. $(x^i, x^j - a_{ij} e_k) \in T(R_v)$ denote by A the matrix of all a_{ij}. Let $g_{ij} = \max\{a_{ij}, 0\} \geq 0$ denote by G matrix of all g_{ij}. Let $b_{ij} \in \mathbb{R}$ be the maximum number s.t. $(x^i, x^j - b_{ij} e_k) \in T(R_v)$ denote by B the matrix of all b_{ij}. Denote by $A - j$ the minor of matrix A without j-th column and row.

Algorithm 1 QLSARP

function QLSARP($C, T(R_v)$)
    ▷ Function Returns 1 if R_v satisfies QLSARP and 0 otherwise
    Check SARP
    if SARP is failed **then**
        return 0
    end if
    for all $x^i \in C$ **do**

[35] The consumption experiment $E = (x_i, B_i)_{i=1}^n$ satisfies the General Axiom of Revealed Preference (**GARP**) if for every integer $m \leq n$ and every $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, $x_{i_{j+1}} \in B_{i_j}$ for $j = 1, \ldots, m-1$ implies $x_{i_1} = x_{i_m}$ or $x_{i_1} \notin B_{i_m} \setminus \partial B_{i_m}$.

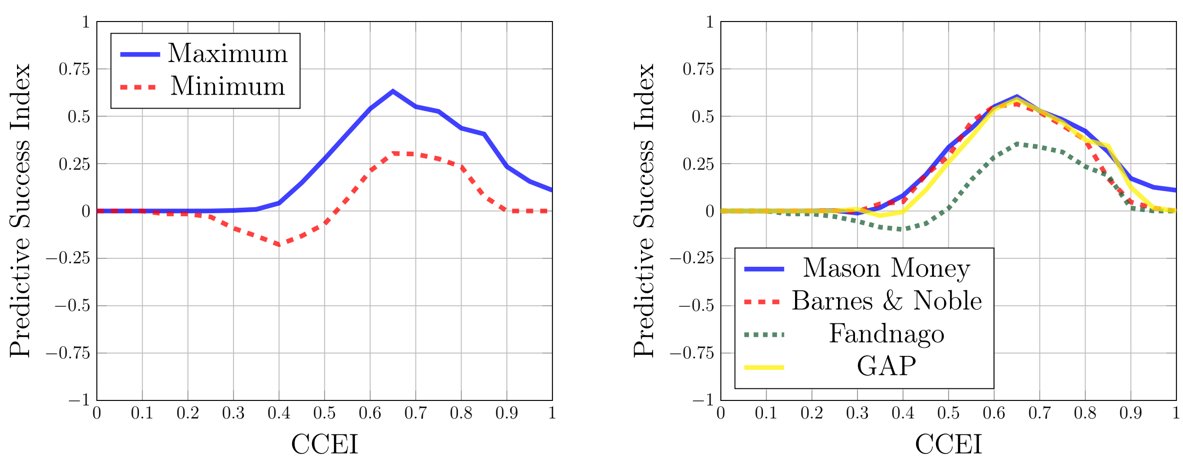
[36] Here and further we are using minimum and maximum instead of infinum and supremum since existence of them is guaranted by compactness of budget sets

```
      for all x^j ∈ C \ {x^i} do
          Define Weighted DiGraph from adjacency matrix G − i
          P ← vector of all shortest paths from x^i to all elements in C \ {x^j}
          for all x^k ∈ C \ {x^i} do
              if a_ij + p_k < b_ki then
                  return 0
              end if
          end for
      end for
  end for
  return 1
end function
```

Using simple Dijkstra algorithm to find shortest paths from one vertex in graph to all others the complexity of algorithm will be $O(n^4)$.

Appendix D: More On Quasi-Linearity in Goods

Let us provide more evidence on quasi-linearity in goods. We start from showing the analysis for quasi-linearity in commodities from the experimental data we collected. Recall that using the data from Mattei (2000) and the survey data from ECPF we got the strong positive evidence for quasi-linearity in goods under marginally larger decision errors levels. However, the both data sets have the feature of not sufficient price variation, unlike the experiment we conducted.



(a) Maximum and Minimum: Individual Numeraire (b) By Commodities: Common Numeraire

Figure 18: Predictive Succes Index

Figure 18 shows the predictive success indexes for the assumption of quasi-linearity in goods. Figure 18(a) shows the predictive success index for the assumption of individual numeraires, that is the numeraire for each person can be different and chosen to either maximize on minimize the predictive success. Figure 18(b) shows the predictive success index for the assumption of common numeraire, that is we suppose each of goods possible to be a numeraire and compute the predictive success index for it. Note that assumption of quasi-linearity in goods (especially assuming the individual numeraire) looks favorable comparing to the assumption of quasi-linearity in money (Figure 10). Note as well, that the CCEI at which predictive success index peaks for the assumption of quasi-linearity in goods is higher comparing to the CCEI at which predictive success peaks for quasi-linearity in money. This reduces, the "costs" of rationality, i.e. to look quasi-linear in money person has to give up 35% of income, while to look quasi-linear in goods only 30%.

The support for the assumption of quasi-linearity in goods in we see less support in the experimental data than in the case of experiment from Mattei (2000) and the ECPF (Figures 8 and 12 respectively). However, for the case of individual numeraires evidence is still supporting the assumption of quasi-linearity, since the lower bound of 95% confidence interval (at CCEI of .7) is still positive (about .15).[37] Therefore, the hypothesis of quasi-linear preferences looks favorable comparing to the null hypothesis of uniform random choices. Note, that comparing to the previous evidence on quasi-linearity in goods we provided, this case requires allowing larger decision making error. This "change" in decision making quality should not be surprising, since if we compare it to the results we obtained from Mattei (2000) data, then we will see that this is due to the change of the power of test. Larger price variation and larger amount of budgets in our experiment allows us to have mean (and median) level of CCEI lower for uniform random subjects.

Note that Figure 18 compares as well the assumption of common versus individual numeraires. Both assumptions get the empirical support from the data, but the 95% confidence interval lower bound for the assumption of individual numeraires is three times of 95% confidence interval lower bound for the assumption of the common numeraire. While, there is no significant difference between the best predictive success for the assumption of common numeraires and the assumption of individual numeraires, assuming that different people can have different numraires still make the assumption of quasi-linearity in goods more convincing.

Appendix E: Experimental Procedures and Instructions

The experiment consisted of 30 independent decision problems. In each of them subjects were asked to allocate tokens among five accounts (commodities): "Cash", "Mason Money",

[37]The confidence interval is estimated using the results from Demuynck (2014).

"Barnes and Noble gift card", "Fandango gift card" and "GAP gift card". Due to the restrictions from the companies there were additional restrictions:

– Minimum positive amount of Mason Money is $5.

– Minimum positive amount of Barnes and Noble gift card is $10

– Minimum positive amount of Gap gift card is $10.

– For Fandango gift card amount could be: $0, $15, $25,$35.

The resolution in each commodity (except Fandango gift card) is $1.

Before starting the experiment instructions were read out aloud and before that each subject received the paper copy of the instructions. All subjects faced the quiz, that was testing their understanding of the decision making task. At the end of experiment one of the decision problems was randomly chosen (from the discrete uniform distribution) and subjects were paid according to the decisions they made in that period. We used e-gift cards, which were sent to subjects official GMU e-mail addresses and were possible to be used right away. Cash choices were as well paid at the end of the experiment, and Mason Money were sent to their Mason Money accounts at the end of the experiment.[38]

Experiment is implemented using oTree (Chen et al. (2016)) and the demo version of the experiment is available here.

[38]Each Student of George Mason has a Mason Money account connected to the Student ID.

INSTRUCTIONS

Thank you for participating in today's experiment. Please remain silent during the experiment. If you have any questions, please raise your hand and the experimenter will assist you in private.

This is an experiment in individual decision-making. Your earnings from the experiment will depend in part on your decisions and on chance. Your earnings will **not** depend on the decisions of other participants. Please pay careful attention to the instructions as a considerable amount of money is at stake.

Your payment in today's experiment will not be lower than $15 equivalent. This will be paid to you at the end of the experiment in private.

You will face 30 decision problems. In each decision problem, you will be given 100 tokens to be divided among 5 commodities. The five commodities are:

- **Cash**

  You can choose any (integer) dollar amount of this commodity. The amount of your choice will be paid to you at the end of experiment.

- **Mason Money**

  This is a prepaid debit program that provides a fast, safe and convenient way to make purchases on and off campus. Mason Money is accepted at all cafes and restaurants on campus and is linked to your Mason ID.

- **Barnes and Noble gift card**

  Barnes and Noble is the largest retail bookseller in the United States, and a leading retailer of content, digital media and educational products in the country.

- **Fandango gift card**

  Fandango allows to buy tickets to more than 26,000 theaters nationwide. It is available online, and through their mobile and connected television apps.

- **Gap gift card**

  Gap is a US-based multinational clothing and accessories retailer. The card can be used in any store or the online store.

You will be asked to allocate your 100 tokens to each of these commodities for 30 different sets of prices for each commodity. Prices will range from 2.5 tokens per dollar to 15 tokens per dollar.

Prices are set up in the way that you can always buy an equivalent of $15. For instance, suppose that each commodity price is 5 tokens per dollar, then you can purchase $20 in cash, or $4 in each commodity.

Due to the restrictions set by companies on gift cards **additional restrictions** apply**:**

- If you purchase any positive amount of Mason Money, you should purchase at least $5.
- If you purchase any positive amount of Barnes and Noble gift card, you should purchase at least $10
- If you purchase any positive amount of Gap gift card, you should purchase at least $10.
- For Fandango gift card, you should purchase one of the following amounts: $0, $15, $25, $35.

Sample Screenshot:

Decision (1 out of 30)

	Prices (tokens per $)	Dollars in Commodity	Tokens in Comodity
Cash	**6.0** tokens per dollar	0	0.0 Tokens
Mason Money	**11.5** tokens per dollar	0	0.0 Tokens
Barnes and Noble gift card	**5.0** tokens per dollar	0	0.0 Tokens
Fandango gift card	**4.0** tokens per dollar	0	0.0 Tokens
Gap gift card	**7.5** tokens per dollar	0	0.0 Tokens
		Your total Expenditure is	0.0

Next

Figure 1. Decision Screen

Figure 1 is an example of the decision screen. The first column contains the list of commodities. The second column contains the list of prices for each commodity (in tokens per dollar). The third column contains the quantity of commodity you will be asked to choose. This is the column you will use to make your decisions. The fourth column shows the allocation of your tokens among commodities ($Quantity\ of\ commodity$*$Price\ of\ commodity$).

Once you have decided what allocation of tokens you prefer, press the button **“Next”**. The computer will present you with the next decision problem or a screen asking you to wait for your final payments if you already went through 30 decision problems.

Your earnings:

Your earnings from the experiment are determined as follows. At the end of experiment, the computer will randomly select one of 30 decision problems. You will be paid according to choices you made. This means that each decision problem has the same 1 in 30 chance of being randomly selected. Note that each decision problem is independent from each other. Therefore, please pay attention to each one of them.

Once the computer randomly selects a decision problem, it will privately show you your choices (in the selected decision problem) and your payment (in terms of dollars of each commodity). As part of the check out process, you will have an opportunity to observe the experimenters imputing the amount of Mason Money and other commodities on the web. Note that e-gift cards purchased today can **only** be sent to your Masonlive e-mail account. You therefore need to provide a valid masonlive account to implement your payments.

Note that we will start conducting payments only after everyone in the room has finished the experiment. If you finish early, please remain silent and wait until everyone is done with the experiment.

If you have any questions, please raise your hand and and an experimenter will assist you in private.

Thank You and Good Luck!

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