MATCHING WITH QUOTAS

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Abstract. We characterize the set of solutions for the many-to-many matching problem with quotas using centralized and decentralized mechanisms. Decentralized mechanism is a simple bargaining game in which courses proposes to the students they want and students choose the most preferred courses from the observed set of proposals. Centralized mechanism is an iterative procedure such that at every stage every agent (meaning course or student) chooses the most preferred set of partners over those who prefer the agent to the current match. Usual conditions on preferences used in the literature can not be applied for the problem with quotas. However, the generalizations of them can be used to together with restrictions on the space of matchings can sucesfully guarantee the characterization of the sets of stable matchings using centralized and decentralized mechanisms.

1 Introduction

This paper deals with many-to-many matching problem with quotas and characterizations of its solutions. Given a set of students and courses, matching is an assignment of (groups) of students to the courses and of (groups) of courses to the students, such that student

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is assigned to the course if and only if course is assigned to the student. In general many-to-many matching markets are not understood as well as many-to-one markets, regardless of important examples of real-world many-to-many matching markets.

The best-known example is probably the market for medical interns in the U.K. (see Sotomayor (1990)). This example is important because it works through a centralized matching mechanism. Another motivating example is actually course admission problem we introduced above. Echenique and Oviedo (2006) use the example of firms and consultants, such that every firm can hire multiple consultants as well as every consultant can be hired by several firms. Two more real-world many-to-many labor market example are the high school teachers in Argentina (around 35% of teachers work for multiple schools) as well as university professors in Russia who also usually hold appointments at multiple universities. Note that even the U.S. around 5% of employees hold multiple jobs.

Note that those problems have quotas. In terms of course admission problem quotas for students are minimum and maximum amount of credits to take and quotas for courses are the minimum amount of students needed to happen and the capacity of the course. Despite its clear practical implications, we know little about matching problems with quotas. We study the case in which each matching assignment is required to satisfy both lower and upper quotas, as in Biró et al. (2010).[1]

Substitutability is a crucial condition for matching theory. Hatfield and Milgrom (2005) show that substitutability guarantees existence of the stable matching with contracts Moreover, it is used to gurantee the non-emptiness of the core(Ostrovsky (2008)), ascending clock auctions (Milgrom and Strulovici (2009)), package auctions (Milgrom (2007)). Substitability and the refinement of it called strong substitability allows to characterize the set stable matchings for the many-to-many matching problem as well (Echenique and Oviedo (2006)). However, preferences in the problem with quotas cannot satisfy substitutability. This can

[1]For the case in which these restrictions can be violated see Fragiadakis et al. (2016).

be illustrated with a simple example. Let c be a course and s_1, s_2 be students. Assume that c prefers s_1 and s_2 together to s_1 and s_2 separately. Substitutability implies, that if some student is chosen from a given set of alternatives, then the same student would be chosen from a subset of original set of alternatives. Hence, the original preferences of c satisfy substitutability. However, if we impose the *lower quota of two* - the course can not operate having less than two students, then c would prefer to stay alone (choose nothing) from the sets of alternatives that contain only s_1 or s_2. This violates substitutability, even though preferences satisfy substitutability if we consider a problem without lower quotas. In the discussion we provide more formal discussion of this and provide a formal proof that preferences in the non-trivial many-to-one matching problem cannot satisfy substitutability.

We provide a characterization of the set of strong stable and setwise stable matchings using both centralized and decentralized mechanisms. Decentralized mechanism is a simple bargaining game in which courses proposes to the students they want and students choose the most preferred courses from the observed set of proposals. Centralized mechanism (T-operator) is an iterative procedure such that at every stage every agent (meaning course or student) chooses the most preferred set of partners over those who prefer the agent to the current match. If preferences of students satisfy generalized substitutability, then the set of equilibria of simple bargaining game equals to the set of strong stable matchings. If in addition to that preferences of courses satisfy generalized strong substitutability, then the set of maximal setwise stable matchings is equal to the set of fixed points of T-operator.

The remainder of this paper is organized as follows. In Section 2 we state the model of many-to-many matching with quotas. In Section 3 we provide a formal definitions of the stability concepts. In Section 4 we provide a formal definition of the centralized mechanism. In Section 5 we define a bargaining game (decentralized mechanism). In Section 6 we show and discuss the results. In Section 7 discuss connection of the results to the previous literature and the applicability of the centralized and decentralized mechanisms for real-world problems.

2 Preliminary Definitions

A matching problem can be specified as a tuple $\Gamma = (N, \mathcal{M}, \mathcal{R})$, where N is the set of players, $\mathcal{M}$ is the set of all possible matchings and $\mathcal{R}$ is the preference profile.

Let us start from defining N, the set of players. For simplicity, we will refer to the two participating sides as students and courses. We use $S = \{s_1, ..., s_n\}$ to denote the set of students and $C = \{c_1, ..., c_m\}$ to denote the set of course, and $N = S \cup C$.

2.1 Mathing Recall that we consider a matching problem with quotas. Therefore, to define $\mathcal{M}$ we need to define quotas first. Let the lower quotas be a function $\underline{q}(a) : A \to \mathbb{N}$ and the upper quotas be a function $\bar{q}(a) : A \to \mathbb{N}$ such that $\bar{q}(a) \geq \underline{q}(a)$ for every $a \in A$ and $A \in \{C, S\}$. Then every agent in a matching either stays unmatched or has between $\underline{q}(a)$ and $\bar{q}(a)$ partners.

An assignment is a correspondence $\nu = (\nu_S, \nu_C)$, where $\nu_S : S \to 2^C$ and $\nu_C : C \to 2^S$. For the simplicity of further notation we can refer to the match of agent a as $\nu(a)$ that would be equal to $\nu_C(a)$ if $a \in C$ or $\nu_S(a)$ if $a \in S$.

Definition 1. *An assignment μ is said to be a* ***matching*** *if :*

(i) *$s \in \mu(c)$ if and only if $c \in \mu(s)$ for every $s \in S$ and $c \in C$,*
(ii) *$|\mu(a)| \in \{0\} \cup [\underline{q}(a), \bar{q}(a)]$ for every $a \in C \cup S$.*

That is matching is an assignment such that if course is matched to a student, then the student is matched to the course. Moreover, we require every agent either to stay unmatched or to have the number of partners that satisfies quotas. There is a particular class of matchings which attracts a lot of interest (at least in graph theory), that is a maximal matchings.

Definition 2. *A matching μ is said to be* ***maximal*** *if $|\mu(a)| \in [\underline{q}(a), \bar{q}(a)]$ for every $a \in C \cup S$.*

That is maximal matching is a matching in which every agent has a number of partners that satisfies quotas. The original definition of maximal matching is that there is no unmatched agents, and we generalized it for the problem with quotas. There would not be unmatched agents if $\underline{q}(a) \geq 1$. The modification we make allows us to consider every matching as maximal if $\underline{q}(a) = 0$. This would be important to establish the clear connection between the results we obtain and the results from the literature. Denote the set of all maximal matchings by $\mathbb{M}$.

2.2 Preferences Now we need to define the preference profile $\mathcal{R}$, the set of preference relations. We assume that every agent $a \in C \cup S$ has a linear order[2] of preferences over possible matches. Let $\tilde{R}(a)$ be a preference relation of a over sets of partners. A match X is said to be **acceptable** by agent $a \in C \cup S$ if $X\tilde{R}(a)\emptyset$, i.e. a match X is preferred to the stay-alone option.[3] Let $R(a)$ be the *truncated preference relation*, such that any $X \neq \{\emptyset\}$ with $|X| < \underline{q}(a)$ or $|X| > \bar{q}(a)$ is unacceptable. Note that if $\tilde{R}(a)$ is a linear order, then $R(a)$ is a linear order as well, since it is a permutation of $\tilde{R}(a)$ that makes several sets of partners unacceptable. Denote by $P(a)$ the strict part of preference relation of $R(a)$, for any $a \in C \cup S$. $\mathcal{R}$ consists of all preference relations and $\mathcal{R}(C)$ denotes all preference relations of courses and $\mathcal{R}(S)$ denotes all preference relations of students.

Example: Let $C = \{c_1\}$, $S = \{s_1, s_2, s_3\}$ and $\underline{q}(c_1) = \bar{q}(c_1) = 2$.

$\tilde{R}(c_1) : \{s_1, s_2, s_3\}P(c_1)\{s_1, s_2\}P(c_1)\{s_1, s_3\}P(c_1)\{s_1\}P(c_1)\emptyset$

(a) Original preference relation

$R(c_1) : \{s_1, s_2\}P(c_1)\{s_1, s_3\}P(c_1)\emptyset$

(b) Truncated preference relation

FIGURE 1. Illustration of truncated preference relation

Figure 1 illustrates the construction of truncated preference relation from the original one. Figure 1(a) shows the original preference relation over sets of alternatives. In this case the set $\{s_1, s_2, s_3\}$ is unacceptable, since it exceeds the capacity (upper quota) and the alternative $\{s_1\}$ is unacceptable since it would not allow to fulfill lower quota. Eliminating these alternatives we arrive to the truncated preference relation shown in Figure 1(b).

[2]A linear order is a complete, anti-symmetric and transitive preference relation

[3]Acceptability is defined with respect to particular preference relation.

Let $Ch_a(X) \subseteq X$ be the most preferred set of agent a.[4] Note that since we assume $R(a)$ to be a linear order, $Ch_a(X)$ is unique. So, $Ch_a(X)$ is the unique subset X' of X, such that $X'P(a)X''$ for any $X'' \subseteq X$. Note that for any $a \in C \cup S$ and if $|X| \leq \underline{q}(a)$ or $|X| \geq \bar{q}(a)$, then $Ch_a(X) = \{\emptyset\}$.[5] Let us characterize some properties of a choice function generated by a linear order which we will use later:

- Ch_a is **idempotent**. That is $Ch_a(Ch_a(X)) = Ch_a(X)$.
- Ch_a is **monotone**. That is for any $X \subseteq X'$, $Ch_a(X')\ R(a)\ Ch_a(X)$.

Let us now introduce the conditions on preferences which are commonly used in the literature without quotas and their generalizations for the problem with quotas. As we mentioned above there are two major conditions used for matching with quotas: substitutability and strong substitutability.

Definition 3. *A preference relation $R(a)$ satisfies **generalized substitutability** if for every $X' \subseteq X$, such that $|X'| \geq \underline{q}(a)$, $x \in Ch_a(X \cup \{x\})$ implies $x \in Ch_a(X' \cup \{x\})$ where x is taken from the set of possible partners.*

Preference relation satisfies generalized substitutability if a partner is among the best preferred set of partners from some set that it would be among the set of most preferred partners for any subset of it.

Definition 4. *A preference relation $R(a)$ satisfies **generalized strong substitutability** if for every $X\tilde{R}(a)X'$, such that $|X'| \geq \underline{q}(a)$ and $|X| \geq \underline{q}(a)$, $x \in Ch_a(X \cup \{x\})$ implies $x \in Ch_a(X' \cup \{x\})$ where x is taken from the set of possible partners.*

Preference relation satisfies generalized substitutability if a partner is among the best preferred set of partners from some set that it would be among the set of most preferred partners for any less preferred set of alternatives of it. The generalization of the properties from usual one is that we require the sets to contain at least $\underline{q}(a)$ elements that makes condition applicable for the problem with quotas.

[4]In this case X is an arbitrary set that includes $\emptyset$ as an element.

[5]This statement is correct since we use the truncated preference relation of agents $a \in C \cup S$.

3 Solution Concepts

Let us now introduce the solution concepts we are going to use. First of them is strong stable matching concept that is stable matching in the many-to-one sense. That is absence of the coalition that consist of a course and set of studets who prefer to be matched to each other rather being matched to their current partners. Second is setwise stable matching, that is core-like concept for many-to-many matching problem. That is a generalization of strong stable matching by allowing to have the collection of courses rather than single one.

3.1 Strong Stable Matching The definition of stable* matching is similar to one from Echenique and Oviedo (2004). This concepts prevents from the blocking the matching by coalition of course and bunch of students and the many-to-one case stable* is equal to the core. This concept is asymmetric, therefore it can be defined alternatively with a student and collection of courses, all of the further results would hold with the respective permutation.

Definition 5. *The pair $(c, D) \in C \times 2^S$ is a **block*** of $\mu \in \mathcal{M}$ if*

(i) $D \cap \mu(c) = \emptyset$;
(ii) $c \in Ch_s(\mu(s) \cup c)$ *for all* $s \in D$;
(iii) $s \in Ch_c(\mu(c) \cup D)$

Definition 6. *A matching $\mu \in \mathcal{M}$ is **stable*** if it is individually rational and there is block* of μ. Denote the set of stable* matchings by S^*.*

Note that importance of investigating such “many-to-one driven” solution concept is caused by the applications of many-to-many problem with quotas. A lot of them primarily have students matched to one course only, but there is a small number of students who are matched to several courses. Requiring a setwise stability in such a framework may be a bit of overshooting.

3.2 Setwise Stable Matching The definition of setwise stable matching is due to Sotomayor (1999). It is stronger than a core and stable* matching.

Definition 7. *The triple* $(C', S', \mu') \in C \times S \times \mathcal{M}$ *is a* ***setwise block*** *of* $\mu \in \mathcal{M}$ *if*

(i) $C' \cup S' \neq \emptyset$;
(ii) $\mu' \setminus \mu \subseteq C' \cup S'$ *for all* $a \in C' \cup S'$;
(iii) $\mu'(a)P(a)\mu(a)$ *for all* $a \in C' \cup S'$;
(iv) $\mu'(a) = Ch_a(\mu'(a))$ *for all* $a \in C' \cup S'$.

Definition 8. *A matching* $\mu \in \mathcal{M}$ *is* ***setwise stable*** *if it is individually rational and there is setwise block of* μ*. Denote the set of pairwise stable matchings by* SW*.*

4 Fixed Point Approach

Now we move on to introducing the T operator. We use the T-operator from Echenique and Oviedo (2006).

Definition 9. *Let* ν *be an assignment, then*

– *Let* $U(c, \nu) = \{s \in S : \ c \in Ch_s(\nu(s) \cup \{c\}\}$ *for any* $c \in C$
– *Let* $V(s, \nu) = \{c \in C : \ s \in Ch_c(\nu(c) \cup \{s\}\}$ *for any* $s \in S$.

Set $U(c, \nu)$ is the set of students that that would include c into their most preferred set from s. Set $V(s, \nu)$ is the set of colleges that would include s into their most preferred set from $\nu_C(c) \cup \{s\}$.

Definition 10. *Now define* $T : \mathcal{V} \to \mathcal{V}$ *by*

$$(Tv)(a) = \begin{cases} Ch_a(U(a, \nu)) \ if \ c \in C \\ Ch_a(V(a, \nu)) \ if \ a \in S \end{cases}$$

The T-operator at every operator makes every agent to choose the best set of partners from the possible set of partners that (weakly) prefer the agent to the current match. A matching μ[6] is said to be a **fixed point** of T if $T\mu = \mu$. Denote the set of fixed points of T by $\mathcal{E}$. Further we refer to T-algorithm that is iterative application of T-operator.

[6]In general this should be an assignment but we directly refer to Echenique and Oviedo (2006) who shown that every fixed point of T-operator is individually rational matching

5 Noncooperative Implementation

Let us describe the game Γ. First, every course proposes a set of partners $\nu_c \subseteq S$. Courses make proposals simultaneously. Second, after observing the proposals of courses each students proposes a set of partners $\xi_s \in C$. Students make the proposals simultaneously. Finally matching μ results by $s \in \mu(c)$ if and only if $s \in \eta_c$ and $c \in \xi_s$.

Definition 11. *A strategy profile (η^*, ξ^*) is a* ***subgame perfect Nash equilibrium*** *(SPNE) of Γ if:*

(i) $\xi^*(\eta) \cap \{c : s \in \eta_c\} R(s) A$ *for every* $A \subseteq \{c : s \in \eta_c\}$*;*
(ii) $\eta^*_c \cap \{s : c \in \xi^*_s(\eta)\} R(c) X \cap \{s : c \in \xi^*_s(X, \eta)\}$ *for every* $X \subseteq S$*.*

Denote by $SPNE$ the set of matching generated by subgame perfect Nash equilibria of Γ.

The Γ is a generalization of the game proposed by Alcalde and Romero-Medina (2000)[7] for the case of many-to-many matching done by Echenique and Oviedo (2006).

6 Results

We can summarize the results from the paper in the Table 1. We move all the proofs to the Appendix.

		$R(S)$		
		Arbitrary	Generalized Substitutability	Generalized Strong Substitutability
$R(C)$	Arbitrary	$SW \subseteq \mathcal{E} \subseteq S^*$ $S^* \subseteq SPNE$	$S^* = SPNE$ $\mathcal{E} \cap \mathbb{M} = S^* \cap \mathbb{M}$	
	Generalized Substitutability			$SW \cap \mathbb{M} = \mathcal{E} \cap \mathbb{M}$

Table 1. Results

Without making any assumptions about preferences we can already tell that $SW \subseteq \mathcal{E} \subseteq S^* \subseteq SPNE$. That is every setwise stable matching is a fixed point of the T-operator, every fixed point of the T-operator

[7]We use the "College-propose-and-student-choose" version of the game, but it may be switched and all of results remain true with the respective permutation in the concept of stable* matching and the properties about preferences.

is stable* matching, and every stable* matching can be an outcome of some SPNE of the bargaining game.

Assuming the generalized substitutability of students' preferences we are getting that matchings generated by a subgame perfect Nash equilbibria of the bargaining game are stable*. Moreover, generalized substitutability of students preferences also guarantees that $\mathcal{E} \cap \mathbb{M} = S^* \cap \mathbb{M} = SPNE \cap \mathbb{M}$ that is the set of all maximal matchings which are fixed points of T-algorithm is equal to the set of all maximal stable* matchings and to the set of maximal matchings that can be generated by an SPNE of the bargaining game.

Assuming generalized strong substitutability of students' preferences and generalized substitutability of courses' preferences we obtain that $\mathcal{E} \cap \mathbb{M} = SW \cap \mathbb{M} = SPNE \cap \mathbb{M}$, that is set of all maximal matchings which are fixed points of T-algorithm is equal to the set of all maximal setwise stable matchings and to the set of maximal matchings that can be generated by an SPNE of the bargaining game.

Note at first that unlike in Echenique and Oviedo (2006) we can not obtain the sufficient condition for the non-emptiness of the core using the existence of fixed points of the T-operator. We can use the simple example from Biró et al. (2010) to show that even when $\mathcal{R}$ is generalized strongly substitutable, there may be no stable* matching. This would imply as well that there is no setwise stable matching.

Example: Let $C = \{c_1, c_2\}$, $S = \{s_1, s_2\}$, $\underline{q}(c_1) = \underline{q}(c_2) = \bar{q}(c_1) = \bar{q}(c_2) = 1$, $\underline{q}(s_1) = \underline{q}(s_2) = \bar{q}(s_1) = \bar{q}(s_2) = 1$.

$$\begin{aligned} R(s_1) &: \ c_1 \succ c_2 \succ \emptyset \\ R(s_2) &: \ c_2 \succ c_1 \succ \emptyset \\ R(c_1) &: \ \{s_1, s_2\} \succ \emptyset \\ R(c_2) &: \ s_1 \succ s_2 \succ \emptyset \end{aligned}$$

Note that this is many-to-one matching problem with only one course having non-trivial quota. Note as well that there still is a pairwise stable matching: $\mu(s_1) = c_1$ and $\mu(s_2) = \emptyset$. There is a block* that is $(c_2, \{s_1, s_2\})$, but since c_2 has a lower quota of 2 there is no blocking pair. Note as well that this example fails the possible of generalization of the sufficient condition for non-emptiness of $\mathcal{E}$ from Echenique and Oviedo (2006). In this case $\mathcal{R}$ satisfies generalized strong substitutability, but there is no fixed points of T-algorithm.[8]

Let us also show that our result under generalized strong substitutability is tight.

Example: Let $C = \{c_1, c_2\}$, $S = \{s_1, s_2\}$, $\underline{q}(c_1) = \underline{q}(c_2) = \bar{q}(c_1) = \bar{q}(c_2) = 2$

$$R(s_1):\ \{c_1, c_2\} \succ \emptyset$$
$$R(s_2):\ \{c_1, c_2\} \succ \emptyset$$
$$R(c_1):\ \{s_1, s_2\} \succ \emptyset$$
$$R(c_2):\ \{s_1, s_2\} \succ \emptyset$$

Note the in this example $\mathcal{R}$ satisfies generalized strong substitutability and the problem has unique setwise stable matching that is $\mu(c_1) = \{s_1, s_2\}$ and $\mu(c_2) = \{s_1, s_2\}$. However, $\mathcal{E}$ contains as well the matching $\mu(c_1) = \mu(c_1) = \emptyset$. Moreover, this matching is stable*, therefore, it is also the outcome of SPNE of the bargaining game.

7 Discussion

In the discussion we address two major points. First, we show that neither substitutability nor strong substitutability are applicable for the case of matching problem with non-trivial quotas. Second, we discuss the applicability of centralized and decentralized mechanisms.

7.1 (Strong) Substitutability Echenique and Oviedo (2006) show that if preferences of all agents satisfy substitutability, then the set of fixed point of T-operator coincides with the set of stable* matchings and the set is non-empty. Let us show that this is inapplicable for the problem with quotas.

Definition 12. *A preference relation $R(a)$ satisfies **substitutability** if for every $X' \subseteq X$ $x \in Ch_a(X)$ implies $x \in Ch_a(X')$ where x is taken from the set of possible partners.*

[8] This immediately follows from the fact that if students have substitutable preferences $S^* = \mathcal{E}$ and S^* being empty.

Lemma 1. *If there is a college $c \in C$, such that $\underline{q}(c) \geq 2$ and c has at least one acceptable set of students, then $\mathcal{R}$ is not substitutable.*

Lemma 1 shows that if there is at least one college quota at least two, then the college's truncated preference relation violates substitutability. Hence, the condition from Echenique and Oviedo (2006) can not be applied for the problem with quotas. Note as well that the results we obtain generalize the result from Echenique and Oviedo (2006). Matching problem without quotas can be defined as $\underline{q}(a) = 0$ for every $a \in C \cup S$. Hence, generalized substituability equivalent to the substitutability and the set of maximal matchings coincide with the set of all matchings.

Another result from Echenique and Oviedo (2006) tells that if preferences of students are substitutable and preferences of courses are strongly substitutable, then the set of fixed points of T-operator coincides with the set of setwise stable matchings.

Definition 13. *A preference relation $R(a)$ satisfies **strong substitutability** if for every $X\tilde{R}(a)X'$, such that $|X'| \geq \underline{q}(a)$, $x \in Ch_a(X \cup \{x\})$ implies $x \in Ch_a(X' \cup \{x\})$ where x is taken from the set of possible partners.*

Note that strong substitutability implies substitutability, hence, it also can not be applied for the problem with notrivial quotas. Similarly, if we relax quotas, then the generalized strong substitutability becomes equivalent to the strong substitutability and the original results from Echenique and Oviedo (2006) holds.

7.2 Computational Complexity Note that T-algorithm can be used to verify in polynomial time whether the matching is stable* or setwise stable. This induce that finding a stable matching is at most NP-hard problem. However, if T-algorithm starts not from a fixed point it may cycle even if there are some fixed points. Hence, finding the core element requires starting from every possible matching - finding a fixed point of T-algorithm is NP-hard problem. Therefore, T-algorithm is not an efficient way to find a solution for the large scale problem.

Note that decentralized mechanism (bargaining game) allows us with minimal assumptions guarantee that every equilibrium outcome of it would be a stable* matching. If one is interested in finding setwise stable matchings decentralized mechanism can guarantee only that maximal matching is setwise stable. However, it is harder to control the outcome of the bargaining game since the solution is decentralized.

Appendix: Proofs

Arbitrary Preferences $\times$ Arbitrary Preferences. First let us list the results from Echenique and Oviedo (2006) since we are not going to prove them, but take them as given. Those who are interested in proofs can proceed to Echenique and Oviedo (2006) paper.

Lemma 2. $SW \subseteq \mathcal{E} \subseteq S^*$

So to complete the first sell we need to prove that $S^* \subseteq SPNE$.

Lemma 3. *$S^* \subseteq SPNE$.*

Proof. Suppose that $\mu \notin SPNE$ and let us show that then $\mu \notin S^*$. One of two conditions can be violated. Let $\xi^*_s = \mu(s)$ and $\eta^*_c = \mu(c)$.

First, assume that there is $X \subseteq \{\bar{c} : s \in \eta_{\bar{c}}\}$ such that $XP(s)\xi^*_s(\eta) \cap \{\bar{c} : s \in \eta_{\bar{c}}\}$, then μ is not individually rational, therefore $\mu \notin S^*$.

Second, assume that there is $X \subseteq S$ such that $X \cap \{s : c \in \xi^*_s(X, \eta^*_{-c})\}P(c)\eta^*_c \cap \{s : c \in \xi^*_s(\eta)\}$. Denote by $D = X \cap \{s : c \in \xi^*_s(X, \eta^*_{-c}\} \setminus \eta^*_c \cap \{s : c \in \xi^*_s(\eta)\}$. Note that since μ is individually rational, $\eta^*_c \cap \{s : c \in \xi^*_s(\eta)\} \cup D$ contains at least $\underline{q}_c$ elements. Then $D \subseteq Ch_c(\mu(c) \cup D)$ as well as for all $s \in D \subseteq S$ $c \in C_s(\mu(s) \cup c)$ because D includes only students from $\{s : c \in \xi^*_s(X, \eta^*_{-c}\}$, that is students who wants to accept the proposal of the course c. Hence (c, D) blocks* μ, hence $\mu \notin S^*$. $\square$

Arbitrary $\times$ Generalized Substitutable Preferences. We need to prove that if preferences are generalized substitutable, then $SPNE \subseteq S^*$ that together with Lemma 3 would imply that $S^* = SPNE$.

Lemma 4. *If $\mathcal{R}(S)$ satisfies generalized substitutability, then $SPNE \subseteq S^*$.*

Proof. On the contrary assume that $\mu \in SPNE$ and $\mu \notin S^*$.

Let (η^*, ξ^*) be an SPNE and let $Y(\eta_{-c}) = \{s : c \in Ch_s(\{c : s \in \eta_c\} \cup c)\}$. Then, (η^*, ξ^*) has to satisfy the following properties:

(1) $\xi_c^*(\eta) \cap \{c : s \in \eta_c\} = Ch_s(\{c : s \in \eta_c\})$
(2) $\eta_c^* \cap Y(\eta_{-c}^*) = Ch_c(Y(\eta_{-c}^*))$.

Let $\mu \in \mathcal{M}$ be an outcome of (η^*, ξ^*).

Let $(\hat{\eta}, \hat{\xi})$ be the pair of strategies obtained from (η^*, ξ^*), by having each s not proposing to courses that did not propose to s and each c not propose to students who will reject. Hence, $\hat{\xi}_s = \xi_c^*(\eta) \cap \{c : s \in \eta_c\}$ and $\hat{\eta}_c = Ch_c(Y(\eta_{-c}^*))$. Let us show that $(\hat{\eta}, \hat{\xi})$ is SPNE as well and its outcome is μ. First t is immediate that its outcome is μ: $\hat{\eta}_c = \mu(c)$ and for all c and for all $s \in \mu(c)$, $c \in \hat{\xi}_s$. To show that $(\hat{\eta}, \hat{\xi})$ is SPNE we need to show that conditions (1) and (2) are satisfied.

(1). Given a strategy profile η for courses, each s is indifferent between proposing to ξ_s^* and $\hat{\xi}_c$ because $\xi_s^* \setminus \hat{\xi}_c$ is the set of courses which would reject s.

(2). We need to show that $\hat{\eta}_c \cap Y(\hat{\eta}_{-c}) = Ch_c(Y(\hat{\eta}_{-c}))$. Note that $s \in Y(\eta_{-c})$ if and only if $c \in Ch_s(\{c : s \in \eta_c\} \cup c)$. If $\mu(s) = \emptyset$ and $\underline{q}(s) \geq 2$, then $Ch_s(\mu(s) \cup c) = Ch_s(\hat{\eta}_c \cup s) = \emptyset$. Hence, $s \notin Y(\hat{\eta}_{-c})$, therefore, $s \notin \hat{\eta}_c \cap Y(\hat{\eta}_{-c})$ and $s \notin Ch_c(Y(\hat{\eta}_{-c}))$. At the same time $\mu(s) = \emptyset$ implies that $s \notin \mu(c) = \hat{\eta}_c$, hence $s \notin Ch_c(Y(\eta_{-c}^*)) = \hat{\eta}_c \cap Y(\eta_{-c}^*)$. Therefore, the unmatched students with lower quotas strictly greater than one can not generate the contradiction.

Consider, $\mu(s) \neq \emptyset$ or $\underline{q}(s) \leq 1$. Note that for this case holds the following equality: $Ch_s(\{c : s \in \eta_c^*\} \cup c) = Ch_s(Ch_s(\{c : s \in \eta_c^*\}) \cup c)$.[9] Since $Ch_s(\{c : s \in \eta_c^*\}) = \mu(s) = \hat{\eta}_s$, $Ch_s(\{c : s \in \eta_c^*\} \cup c) = Ch_s(\{c : s \in \hat{\eta}_c\} \cup c)$. Hence, $Y(\eta_{-c}^*) = Y(\hat{\eta}_{-c})$. Therefore, $\hat{\eta}_c \cap Y(\hat{\eta}_{-c}) = Ch_c(Y(\hat{\eta}_{-c}))$.

Now, let us show that if $\mu \notin S^*$, then $(\hat{\eta}, \hat{\xi})$ is not SPNE. Assume that $\mu \notin S^*$, then there is (c, D) that blocks* μ, that is

(i) $D \cap \mu(c) = \emptyset$;
(ii) $c \in Ch_s(\mu(s) \cup c)$;

[9]This property is a consequence of generalized substitutability, see Echenique and Oviedo (2006) and Blair (1988).

(iii) $D \subseteq Ch_c(\mu(c) \cup D)$.

Condition (ii) implies that $D \subseteq Y(\eta_{-c})$, since $Y(\eta_{-c})$ is a set of students that would prefer to include course c in their current match. Condition (iii) and $D \subseteq Y(\eta_{-c})$ imply that $D \subseteq Ch_c(Y(\eta_{-c}))$. Condition (i) implies that $D \cap \mu(c) = \emptyset$. Since $\mu(c) = \hat{\eta}_c$, $D \cap \hat{\eta}_c = \emptyset$. If $D \subseteq Ch_c(Y(\eta_{-c}))$ and $D \cap \hat{\eta}_c = \emptyset$, then $\hat{\eta}_c \cap Y(\eta_{-c} \neq Ch_c(Y(\eta_{-c}))$. Hence $\hat{\eta}_c$ is not the SPNE strategy, that is a contradiction. □

We need to prove that if preferences are generalized substitutable, then $SPNE \cap \mathbb{M} \subseteq \mathcal{E} \cap \mathbb{M}$ that together with Lemma 3 and Lemma 2 would imply that $\mathcal{E} \cap \mathbb{M} = S^* \cap \mathbb{M}$.

Lemma 5. *If $\mathcal{R}(S)$ satisfies generalized substitutability, then $SPNE \cap \mathbb{M} \subseteq \mathcal{E} \cap \mathbb{M}$.*

Proof. Let (η^*, ξ^*) be an SPNE that generates maximum matching and let $Y(\eta_{-c}) = \{s : c \in Ch_s(\{c : s \in \eta_c\} \cup c)\}$. Similarly to the first part of the proof of Lemma 4 the following pair of strategies would also be an SPNE. Let $(\hat{\eta}, \hat{\xi})$ be the pair of strategies obtained from (η^*, ξ^*), by having each s not proposing to courses that did not propose to s and each c not propose to students who will reject.

Now let us show that $\mu \in \mathcal{E}$. Let $c \in C$ and recall that $Y(\eta_{-c}) = U(c, \mu)$. By the definition of $\hat{\eta}_c$, $\mu(c) = \hat{\eta}_c = Ch_c(U(c, \mu))$.

Let $s \in S$ and take $c \in \mu(s)$. Hence, $s \in \mu(c) = \hat{\eta}_c = Ch_c(Y(\hat{\eta}_{-c}))$. Therefore, $\hat{\eta}_c = Ch_c(\hat{\eta}_c) = Ch_c(\mu(c) \cup \{s\})$, so $c \in V(s, \mu)$. This proves that $\mu(s) \subseteq V(s, \mu)$.

Let us show now that $Ch_s(V(s, \mu)) \subseteq \mu(s)$. Let $c \in Ch_s(V(s, \mu))$, then $\mu(s) \cup \{c\} \subseteq V(s, \mu)$. Recall that μ is maximal matching, hence, $\mu(s)$ contains at least $\underline{q}(s)$ elements. Hence, by generalized substitutability $c \in Ch_s(\mu(s) \cup \{c\})$. Therefore, $s \in U(c, \mu)$. On the contrary assume that $c \notin \mu(s)$ this implies that $s \notin \mu(c)$. The fact that $c \in Ch_s(\mu(s) \cup \{c\})$ implies that $\mu(c) \cup \{c\} P(s) \mu(c)$. But $s \in U(c, \mu)$, then $\mu(c) \cup \{s\} P(c) \mu(c)$ contradicts $\mu(c) = \hat{\eta}_c = Ch_c(U(c, \mu))$. This completes the proof that $\mu(s) = Ch_s(V(s, \mu))$.

Therefore, for every $s \in S$ $\mu(s) = Ch_s(V(s, \mu))$ and for every $c \in C$ $\mu(c) = Ch_c(U(c, \mu))$ that is a definition of fixed point of T-operator. Hence, $\mu \in \mathcal{E}$. □

Note that we can relax the conditions in Lemma 5 and not to require μ to be maximal matching. It is enough for μ to be maximal for students, that is there is no unmatched students.

Generalized Substitutable Preferences $\times$ Generalized Strongly Substitutable Preferences. It is enough to prove that $\mathcal{E} \cap \mathbb{M} \subseteq SW \cap \mathbb{M}$. From Lemma 2 we know that $SW \subseteq \mathcal{E}$, hence $SW \cap \mathbb{M} \subseteq \mathcal{E} \cap \mathbb{M}$. Since $\mathcal{R}(C)$ satisfies generalized strong substitutability, then it satisfies generalized substitutability. Hence, $\mathcal{E} \cap \mathbb{M} = S^* \cap \mathbb{M}$ and $S^* = SPNE$. Therefore, $\mathcal{E} \cap \mathbb{M} = SPNE \cap \mathbb{M}$.

Before proving the following Lemma let us recall one more result from Echenique and Oviedo (2006).

Lemma 6 (Lemma 11.4 in Echenique and Oviedo (2006))**.** *If $\nu \in \mathcal{E}$, then ν is individually rational matching.*

Lemma 7. *If $\mathcal{R}(S)$ satisfies generalized strong substitutability and $\mathcal{R}(C)$ satisfies generalized substitutability, then $\mathcal{E} \cap \mathbb{M} \subseteq SW \cap \mathbb{M}$*

Proof. Let $\mu \in \mathcal{E} \cap \mathbb{M}$, then by Lemma 6 μ is individually rational matching. On the contrary assume that $\mu \notin SW \cap \mathbb{M}$. By definition μ is maximal matching, so let us show that μ not being in $SW \cap \mathbb{M}$ implies that there is a setwise block. Let (C', S', μ') be a setwise block of μ.

Fix $c \in C'$, then $\mu'(c)P(c)\mu(c)$. From μ being individually rational and $\mu'(c)P(c)\mu(c)$ we can infer that $Ch_c(\mu(c) \cup \mu'(c)) \nsubseteq \mu(c)$.

Fix $s \in (Ch_c(\mu(c) \cup \mu'(c))) \cap (\mu'(c) \setminus \mu(c))$. Note that μ is maximal matching, hence $\mu(c)$ has at least $\underline{q}(c)$ elements. Since $\mu'(c)P(c)\mu(c)$, $\mu'(c)$ has at least $\underline{q}(c)$ elements. Then by generalized substitutability of $P(c)$ $s \in Ch_c(\mu(c) \cup \{s\})$, that implies $c \in V(s, \mu)$.

On other hand $s \in \mu'(c) \setminus \mu(c)$ implies that $s \in S'$, then $\mu'(s)P(s)\mu(s)$. Recal that (C', S', μ') is a setwise block, so $\mu'(s) = Ch_s(\mu'(c))$. Further μ' is a matching, then $c \in \mu'(s)$. Recall that μ is maximal matching - $\mu(s)$ contains at least $\underline{q}(s)$ elements, and $\mu'(s)P(s)\mu(s)$ implies that $\mu'(s)$ contains at least $\underline{q}(s)$. Then $s \in Ch_s(\mu'(s) \cup \{c\})$ and by generalized strong substitutability of $R(s)$, $s \in Ch_s(\mu(s) \cup \{c\})$.

Recall that $\mu \in \mathcal{E}$ implies that $\mu(s) = Ch_c(V(s, \mu))$. However, we just shown that $\mu(s) \cup \{c\} \subseteq V(s, \mu)$ and $f \in Ch_c(V(s, \mu)) \setminus \mu(s)$, that is a contradiction. $\square$

Note that the Lemma 7 would hold for $\mathcal{R}(S)$ satisfies generalized substitutability and $\mathcal{R}(C)$ satisfies generalized strong substitutability as well.

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