A Representation Theorem for General Revealed Preference

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Abstract

Following Richter (1966), we provide criteria under which a preference relation implied by a finite set of choice observations has a complete extension that can in turn be represented by a utility function. These criteria rely on a mapping over preference relations, the rational closure, which is a generalization of the transitive closure and is employed to construct the complete extension. We illustrate this approach by revisiting the problem of rationalizing incomplete preferences revealed by a sequence of consumption decisions obtained from different budget sets. Our result relaxes the usual assumptions about the consumption space and the structure of budgets generating the observed choices, and allows for a new interpretation of classical revealed preference axioms.

1 Introduction

Rational behavior is commonly modeled in economics in three different ways. A long tradition, going back to the founders of neoclassical economics if not even earlier,[1] describes rational behavior as the maximization of an objective (utility) function. Another approach, pioneered by Frisch (1926) and developed and popularized by Debreu (1954), identifies rational behavior with the existence of a complete and transitive binary (preference) relation over the objects of choice. A third strand, pioneered by Samuelson (1938), describes rational behavior as the satisfaction of congruence (revealed preference) conditions on finite sets of observed choices.

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[1]See e.g. Stigler (1950) for a historical summary.

The connection between the different approaches to rational behavior has been the object of attention of a theoretical literature in economics starting with the contributions of Debreu (1954) on the problem of representing preference relation by means of utility functions and Afriat (1967) on the problem of the construction of utility functions on the basis of finite data sets. A seminal contribution by Richter (1966) considers the connection between the three different approaches, providing a general equivalence result between congruence conditions on finite sets of observed choices and the existence of preference relations, and an equivalence result between congruence conditions on sets of observed choices from competitive (linear) budgets and the representation of the underlying preferences by means of a utility function. Further efforts to connect these notions have been taken by Jaffray (1975) and Bossert et al. (2002) in terms of constructing an upper semi-continuous utility-representable extension of a given preference relation.

In this paper we seek to connect the three models of rational behavior in a parsimonious way. That is, we seek for criteria under which a preference relation implied by a finite set of choice observations has a complete extension that can in turn be represented by a utility function. To this end, we build on the functional approach of Duggan (1999) and Demuynck (2009), and introduce the notion of a rational closure as a mapping from (possibly incomplete) preference relations to (possibly incomplete) preference relations whose fixed points are transitive and that preserves separability properties of the original preference relation. We show that the transitive closure considered previously by Duggan (1999) and Demuynck (2009) and others is an example (not unique) of a rational closure.

Our main result is a representation theorem. We show that an incomplete preference relation has a complete, utility-representable extension that is a fixed point of the rational closure if and only if a simple set-theoretic consistency requirement between the rational closure and the incomplete preference relation is fulfilled. Intuitively, we think of the original preference relation as the information on preferences that has been obtained from (not necessarily finite) choice observations. The consistency requirement is then a general congruence condition guaranteeing that the observed behavior can be represented by a utility function. The fact that existence is obtained as a fixed point of a particular mapping can be exploited to obtain desirable properties of the utility function, as discussed below.

We then consider a revealed preference experiment. Intuitively, a revealed preference experiment represents a situation in which information on strict preferences has been obtained from finite, consecutive choice observations. With no restrictions on budget sets or the consumption space, we show that a revealed preference experiment can be rationalized by a utility function if and only if two conditions are satisfied: (1) the different observations do not directly contradict each other, and (2) the consistency requirement identified in the main theorem is satisfied by the union of the consecutive observations with respect to the transitive closure. In our formulation, condition (1) is equivalent to a general version of WARP, and conditions (1) and (2) are jointly equivalent to a general version of SARP. As a corollary, without restrictions on budget sets or the consumption space, a general version of

SARP is necessary and sufficient for a revealed preference experiment to be rationalized by a utility function.

Similarly, we show that a revealed preference experiment can be rationalized by a strictly increasing utility function if and only if conditions similar to (1) and (2) above are satisfied, with the monotone closure (a mapping that incorporates both transitivity and monotonicity criteria) substituting for the transitive closure. This illustrate the point that the techniques in the paper can be put to us to provide tests for the existence of utility functions representing choice observations that satisfy additional properties. An exception is continuity, which is not compatible with the properties of a rational closure–mappings that satisfy continuity do not induce transitivity and lead to inferences on preferences over pairs of alternatives that cannot be obtained from information about a finite sample of preferences over pairs of alternatives.

We also observe that a rationalization by a strictly increasing utility function while allowing for observed choices to be indifferent to some other alternatives in the budget set, as in Varian (1982), requires only a minor relaxation of condition (1). As a corollary a general version of GARP is necessary and sufficient for a revealed preference experiment to be rationalized allowing for indifferences of observed choices by a strictly increasing utility function.

The connections between the different approaches to rational behavior have been an object of attention of the literature for a long time. As mentioned above, the general connection between utility functions and preference relations was originally studied[2] by Debreu (1954) in the context of continuous utility functions. Rader (1963) and Jaffray (1975) relaxed the assumption of continuity and obtained semi-continuous utility rationalization results that were generalized by Bosi and Mehta (2002). Peleg (1970) shown the sufficient condition for existence of a continuous utility representation for incomplete preference relation. More recently Ok (2002), Evren and Ok (2011) have investigated a problem of existence of a vector-valued utility representation of preference relations.

The basic result connecting the set of choices and preference relations was proven by Szpilrajn (1930). Szpilrajn shown that any acyclic preference relation has a complete and transitive extension. Demuynck (2009) generalized the result by providing a condition to test for existence of complete extension that has properties usually assumed by economists.

The connection between finite data sets and the utility functions was originally investigated by Afriat (1967). Afriat (1967) uses linear budgets in a Euclidean consumption space and obtains the existence of a concave, monotone and continuous[3] utility function that is congruous with the observations. Subsequent literature has constructed tests for consistency of the finite consumption data with various utility maximization hypothesis. Kannai (1977), Matzkin (1991), Matzkin and Richter (1991) and Forges and Minelli (2009) address the ques-

[2]Earlier representation theorems under additional assumptions were provided by Cantor (1895) for completely ordered sets and Von Neumann and Morgenstern (1947) for lotteries.

[3]That is, with respect to the usual Euclidean topology.

tion of testing concavity of the utility representation. Varian (1983), Diewert and Parkan (1985), Echenique and Saito (2015) and Polisson et al. (2015) develop tests for separable utility representations including the context of choices under uncertainty. Crawford (2010) propose a test for habit formation models. Reny (2015) extends Afriat (1967) results using linear budgets to infinite data sets–as in our case, the extra generality implies giving up on deriving continuity from the data. Chambers et al. (2010) characterize the testable implications of revealed preference theory. Echenique and Chambers (2016) provide a general, systematic overview of revealed preference results.

The remainder of this paper is organized as follows. In Section 2 we introduce basic definitions on preference extensions and closure mappings. In Section 3 we state and prove our main utility representation theorem. In Section 4 we revisit revealed preference theory from the viewpoint of extensions and closures. In Section 5 we revisit generalized revealed preference from the same viewpoint. In Section 6 we offer concluding remarks.

2 Preliminaries

2.1 Alternatives

We consider an arbitrary set of alternatives X, with elements denoted x, y, etc. Some examples of interest are (a) a finite set, representing job offers available to a worker, houses available to a buyer, etc., (b) the positive orthant of a finite Euclidean space $\mathbb{R}^m_+$, representing bundles of m commodities, (c) the set of sequences $(c_0, c_1, \ldots, c_t, \ldots)$ where $c_t \in \mathbb{R}^m_+$, representing consumption plans potentially available to a long lived agent, and (d) the set of probability measures over $\mathbb{R}$, representing lotteries with monetary rewards or losses.

We introduce additional structure on the set of alternatives as needed. In particular, in order to define continuous preferences, we let (X, τ) be a topological space for some topology τ.[4] The classical continuous representation results of Debreu (1954) and Rader (1963) relay on the topological space (X, τ) being second countable, that is, having a countable base.[5] Continuity is of course an attractive property when there are infinite alternatives as in examples (b), (c) and (d). The Euclidean space of example (b), equipped with the usual Euclidean topology, is a second countable space. There are different topological spaces of interest for examples (c) and (d), and not all of them are second countable; see e.g. Mas-Colell (1986) and Stokey and Lucas (1989). This illustrates the usefulness of representation results for arbitrary sets of alternatives.

[4]Recall that a topology on X is a collection of subsets of X, called open sets, that includes $\emptyset$ and X, and that is closed under arbitrary unions and finite intersections.

[5]Recall that a base for a topology τ on X is a collection $\mathbb{B}$ of open sets, such that every $x \in X$ and every open set U containing x, there is $V \in \mathbb{B}$ such that $x \in V \subseteq U$.

2.2 Preference Relations

A set $R \subseteq X \times X$ is said to be a preference relation. We denote the set of all preference relations on X by $\mathcal{R}$. We denote the inverse relation $R^{-1} = \{(x, y) | (y, x) \in R\}$. We denote the symmetric (indifferent) part of R by $I(R) = R \cap R^{-1}$ and the asymmetric (strict) part by $P(R) = R \setminus I(R)$. We denote the incomparable part by $N(R) = X \times X \setminus (R \cup R^{-1})$.

Definition 1. *Given a preference relation and any alternative in X, the **lower contour set** and the **upper contour set** of x are, respectively,*

$$L_R(x) = \{y | (x, y) \in P(R)\} \quad \text{and} \quad U_R(x) = \{y | (y, x) \in P(R)\}.$$

We list below some properties of a preference relation:

Definition 2. *A preference relation R is said to be*

- ***complete*** *if $(x, y) \in R \cup R^{-1}$ for all $x, y \in X$ (or equivalently $N(R) = \emptyset$).*
- ***transitive*** *if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all $x, y, z \in X$.*
- *Z**-separable** for given $Z \subseteq X$ if for any $(x, y) \in P(R)$ there is $z \in Z$ such that $(x, z) \in R$ and $(z, y) \in R$.*
- ***upper semi-continuous*** *if (X, τ) is a topological space and $L_R(x)$ for all $x \in X$ are open.*
- ***continuous*** *if (X, τ) is a topological space and $L_R(x)$ and $U_R(x)$ for all $x \in X$ are open.*

Completeness and transitivity are the usual desirable properties of preference relations. Separability and continuity play a key role in classical representation results to which we appeal later on.

Examples. Suppose that $X = \mathbb{R}^m_+$, with the usual Euclidean topology. **(a)** It is well known that if R is complete, transitive and continuous, then R is $\mathbb{Q}^m_+$-separable, that is for any $(x, y) \in P(R)$ there is a bundle z all whose components are rational numbers such that $(x, z) \in R$ and $(z, y) \in R$ (see e.g. Kreps (2012), Proposition 1.15). Since $\mathbb{Q}^m_+$ is countable, it follows that R is separable with respect to any collection of subsets of $\mathbb{R}$ that includes $\mathbb{Q}^m_+$. **(b)** Denote by L the lexicographic preference relation, i.e. $(x, y) \in L$ if there is $k \in \{1, \ldots, m\}$ such that $x_i = y_i$ for $i < k$ and $x_k > y_k$. It is well-known that this relation is not $\mathbb{Q}^m_+$-separable (see e.g. Mas-Colell et al. (1995), Example 3.C.1). However, L is $\mathbb{R}^m_+$-separable.

A driving idea in this paper is to extend incomplete preference relations including additional comparisons of pairs of alternatives while preserving the asymmetric part of the original preference relation:

Definition 3. *A preference relation R' is an* ***extension*** *of R, denoted $R \preceq R'$, if $R \subseteq R'$ and $P(R) \subseteq P(R')$.*

2.3 Functions over Preference Relations

In this section we consider general functions $F : \mathcal{R} \rightarrow \mathcal{R}$ defined over the set of preference relations which may be used to extend an incomplete preference relation.

Definition 4. *For any given function $F : \mathcal{R} \rightarrow \mathcal{R}$, we let*

- $\mathcal{R}_F = \{R \in \mathcal{R} | R \preceq F(R)\}$,
- $\mathcal{R}_F^Z = \{R \in \mathcal{R}$ *and R is Z-separable* $|R \preceq F(R)\}$,

$\mathcal{R}_F$ and $\mathcal{R}_F^Z$ are different sets of preference relations that are extended by F.

We list below some properties of a function over the set of preference relations:

Definition 5. *A function $F : \mathcal{R} \rightarrow \mathcal{R}$ is said to be*

- ***monotone*** *if for all $R, R' \in \mathcal{R}$, if $R \subseteq R'$, then $F(R) \subseteq F(R')$,*
- ***closed*** *if for all $R \in \mathcal{R}$, $R \subseteq F(R)$,*
- ***idempotent*** *if for all $R \in \mathcal{R}$, $F(F(R)) = F(R)$,*
- ***algebraic*** *if for all $R \in \mathcal{R}$ and all $(x, y) \in F(R)$, there is a finite relation $R' \subseteq R$ such that $(x, y) \in F(R')$,*
- ***expansive*** *if for any $R = F(R)$ and $N(R) \neq \emptyset$, there is a nonempty set $S \subseteq N(R)$ such that $R \cup S \in \mathcal{R}_F$ and $P(R) = P(R \cup S)$,*
- ***transitive-inducing*** *if any preference relation satisfying $R = F(R)$ is transitive,*
- ***separability-preserving*** *if there is a countable set Q_F such that for any countable set Z and $R \in \mathcal{R}_F^{Q_F \cup Z}$, $F(R)$ is $(Q_F \cup Z)$-separable.*
- ***upper semi-continuous*** *if (X, τ) is a topological space and $R \in \hat{\mathcal{R}}_F$ implies $F(R)$ is upper semi-continuous,*
- ***continuous*** *if (X, τ) is a topological space and $R \in \tilde{\mathcal{R}}_F$ implies $F(R)$ is continuous.*

Any function $F : \mathcal{R} \rightarrow \mathcal{R}$ that is monotone, closed and idempotent is called a **closure**. A closure is algebraic as defined above if any element of the closure can be obtained from applying the closure to a finite subset of the original relation.[6]

[6] See e.g. Davey and Priestley (2002), definition 7.12.

Expansiveness and transitivity impose conditions on fixed points of F. Expansiveness of F, in particular, means that we can add some indifference pairs to any fixed point $R = F(R)$ that is not complete, such that the new relation will be in $\mathcal{R}_F$.

Separability preserving implies that if R is separable with respect to *any* countable set Z and is extended by F, then we can augment Z so that F preserves separability with respect to the augmented set. This is because $R \in \mathcal{R}_F^Z$ implies $R \in \mathcal{R}_F^{Q_F \cup Z}$.

Gathering the first seven properties, we can define the following:

Definition 6. *A function $F : \mathcal{R} \to \mathcal{R}$ is said to be a* ***rational closure*** *if it is an expansive algebraic closure that induces transitivity and preserves separability.*

Intuitively, a rational closure is a rule which may be useful to extend some original incomplete preference relation, and that satisfies certain desirable criteria. Those criteria include not losing information contained in the original preference relation (closeness and monotonicity), being thorough in using that information (idempotence), using finite sets of information contained in the original preference relation to make each binary comparison (algebraicity), being able to incorporate some indifferences (expansiveness), and inducing transitivity and preserving separability of the original preference relation, both of which are useful to build a utility representation.

The **transitive closure** provides a natural example of a function over preference relations. Denote it by

$$T : \mathcal{R} \to \mathcal{R},$$

where $(x, y) \in T(R)$ if and only if there is a finite sequence $s_1, \dots, s_n$ such that $(s_j, s_{j+1}) \in R$ for every $j = 1, \dots, n-1$, and $s_1 = x$ and $s_n = y$.

We claim:

Lemma 1. *The transitive closure $T : \mathcal{R} \to \mathcal{R}$ is a rational closure.*

Proof. It is easy to check that T is an algebraic closure and that it induces transitivity.

To prove that T is separability-preserving, recall that by definition for any $(x, y) \in T(R)$ if and only if there is a finite sequence $s_1, s_2, \dots, s_n$ such that $(s_j, s_{j+1}) \in R$ for every $j = 1, \dots, n-1$, and $s_1 = x$ and $s_n = y$. This implies that for any $(x, y) \in P(T(R))$ there is some $k \in \{1, \dots, n-1\}$ such that $(s_k, s_{k+1}) \in P(R)$. Now suppose the R is Z-separable; this implies that there is some $z \in Z$ such that $(s_k, z), (z, s_{k+1}) \in R$. But then $(x, z), (z, y) \in T(R)$. Thus, $T(R)$ is also Z-separable. (That is, in terms of the definition or separability-preservation, $Q_T = \emptyset$.)

To prove that T is expansive, consider a relation $R = T(R)$ and assume that $N(R) \neq \emptyset$. Take any element $(x, y) \in N(R)$ and consider the relation $R' = R \cup \{(x, y), (y, x)\}$. We claim that $R' \preceq T(R')$, which would prove that T is expansive. It is clear that $R' \subseteq T(R')$. Therefore, we only need to show that $P(R') \subseteq P(T(R'))$. Assume, on the contrary, that there are elements z and w for which $(z, w) \in P(R')$ and $(w, z) \in T(R')$, and note that

$(x, y) \neq (z, w) \neq (y, x)$. From the definition of T, we know that there is some finite sequence $s_1, \ldots, s_n$ such that $s_1 = w$, $s_n = z$, and $(s_j, s_j + 1) \in R'$ for each $j = 1, \ldots, n-1$. Let m be the minimal integer such that there is such sequence of length m, and let S be any such sequence of length m.

Given a sequence S as described above, there is some j such that either $(s_j, s_{j+1}) = (x, y)$ or $(s_j, s_{j+1}) = (y, x)$ for some $1 < j < m-1$; otherwise $(w, z) \in T(R) = R$, contradicting $(z, w) \in P(R')$. Suppose without loss of generality that $(s_j, s_{j+1}) = (x, y)$ for some $1 < j < m-1$; then there is no $k \neq j$ such that $(s_k, s_{k+1}) = (y, x)$ or $(s_k, s_{k+1}) = (x, y)$, otherwise S would not be the shortest sequence from w to z such that every consecutive pair is in R'. Since $(z, w) \in P(R')$, we have $(z, w) \in R'$. Now consider the finite sequence $y, s_{j+2}, \ldots, s_{m-1}, z, w, s_1, \ldots, s_{j-1}, x$. Note that every pair of consecutive elements of the sequence is in R' and is different from (x, y) and (y, x), so every pair of consecutive elements of the sequence is in R. But then $(y, x) \in T(R) = R$, contradicting $(x, y) \in N(R)$. □

Note that separability-preserving holds for the transitive closure in a very simple form. That is, if R is Z-separable for any Z and extended by T, then $T(R)$ is also Z-separable. We are giving more latitude in the definition of a separability-preserving function to accommodate other useful rational closures. In particular, the monotone closure, described in Section 4, requires to expand Z judiciously in order for the monotone closure to preserve separability with respect to the augmented set.

It is simple to check that the transitive closure is not upper semi-continuous and therefore not continuous for arbitrary topological space (X, τ). As an illustration of a closure that is continuous–and therefore, a fortiori upper semi-continuous–for arbitrary topological space (X, τ), consider the **continuity closure** given by

$$C : \mathcal{R} \to \mathcal{R},$$

where $(x, y) \in C(R)$ if and only if there is a sequence $(x_n, y_n) \in R$, such that $x_n \to x$ and $y_n \to y$. Unfortunately, the continuity closure is neither algebraic nor transitive, as shown by means of examples below.

Examples. (a) Consider the preference relation R' over $X = \mathbb{R}$, equipped with the Euclidean topology, such that $(x, y) \in I(R')$ if and only if either $x, y > 1$, or $x, y < 1$, or $x, y = 0, 1$, and $(x, y) \in P(R')$ if and only $x > 1 > y$. We can check that $(x, y) \in I(T(R'))$ if and only if either $x, y > 1$ or $x, y \leq 1$ and $(x, y) \in P(T(R'))$ if and only $x > 1 \geq y$. Note that $L_{R'}(x)$ is open for $x > 1$, but $L_{T(R')}(x)$ is not. That is, $T(R')$ is transitive and extends R', but it is not upper semi-continuous. (See Figure 1.) We can also check that $(x, y) \in I(C(R'))$ if and only if $x, y \geq 1$ or $x, y \leq 1$, and $(x, y) \in P(C(R'))$ if and only if $x > 1 > y$. That is, $C(R')$ is continuous and extends R', but it is not transitive.

(b) Consider the preference relation $R'' = \{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$ over $X = \mathbb{R}$, equipped with the Euclidean topology. Note that $(0, 1) \in C(R'')$, but there is no finite sub-relation $R \subset R''$ such that $(0, 1) \in C(R)$, showing that the continuous closure is not algebraic.

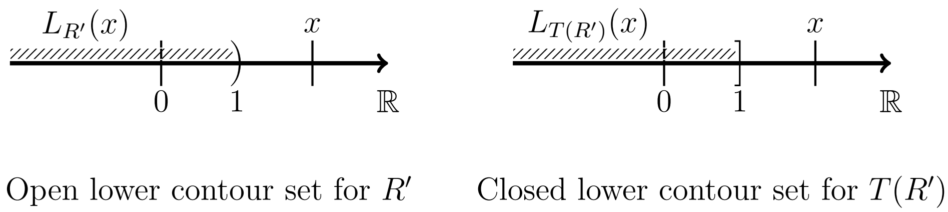


Figure 1: Upper semi-continuous R' but not upper semi-continuous $T(R')$

A condition to apply the classical representation results of Debreu (1954) and Rader (1963) is that (X, τ) is a second countable topological space for some topology such that contour sets are open. If (X, τ) is second countable, in fact, upper semi-continuity of F implies that F is separability-preserving with respect to the base of the topology. That is, for a given second countable topological space, separability preserving with respect to the base of the topology, upper semi-continuity, and continuity are increasingly demanding conditions on F. However, as illustrated by the discussion above, upper semi-continuity is in fact too demanding to be of interest for our approach as it conflicts with other desirable properties.

2.4 Consistency

As shown below, the following is a necessary and sufficient condition for $F(R)$ to be an extension of R:

Definition 7. *Given a function $F : \mathfrak{R} \to \mathfrak{R}$, a preference relation R is said to be **F-consistent** if $F(R) \cap P^{-1}(R) = \emptyset$.*

Example. Let the set of alternatives be $X = \{x_1, x_2, x_3, x_4, x_5\}$ and consider the preference relation $R = \{(x_1, x_2), (x_2, x_3), (x_3, x_1)\}$. This relation is not transitive and is not T-consistent (see Figure 2) because $(x_1, x_3) \in T(R)$ and $(x_3, x_1) \in P(R)$. On other hand $R' = \{(x_1, x_2), (x_2, x_3), (x_4, x_5)\}$ is not transitive but it is T-consistent. Note that transitivity of R is sufficient but not necessary for T-consistency of R.

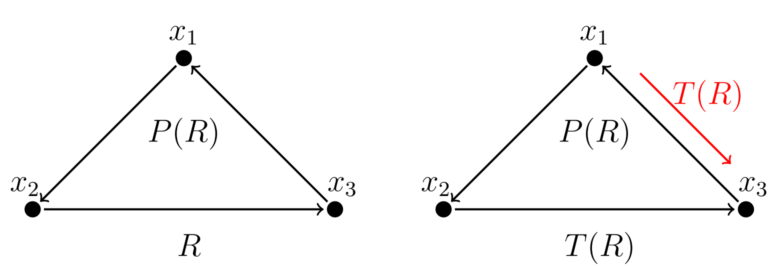


Figure 2: Violation of T-consistency

From idempotence and transitivity-preserving it follows that, if F is a rational closure, then $F(R)$ is transitive for any R. Thus, if F is a rational closure, $F(R) \supseteq T(R)$, so that

F-consistency implies T-consistency. That is, transitivity is a minimum requirement for an extension rule intended to lead to a utility representation, but F may incorporate other desiderata.

3 A Representation Theorem

Our main result is a theorem providing conditions for the existence of a utility function that represents the complete extension of a given preference relation. We do it by showing the existence of a complete relation that is a fixed point of a rational closure. Following Debreu (1954), we define as a **natural topology** for a given complete and transitive preference relation R any topology such that R is continuous.

Theorem 1. *Let F be a rational closure and let $R \in \mathcal{R}$ be Z-separable for some countable set Z. R has a complete extension $R^* = F(R^*)$ that can be represented by a utility function if and only if R is F-consistent. Moreover, the utility function is continuous in any natural topology.*

To prove Theorem 1 we need several supplementary results. We use the following result (contained in Lemma 1 in Demuynck (2009)) repeatedly in the proofs:

Lemma 2. *If $F : \mathcal{R} \to \mathcal{R}$ is closed, then $R \in \mathcal{R}_F$ if and only if R is F-consistent.*

Proof. Since $R \subseteq F(R)$ by assumption, we only need to show that $P(R) \subseteq P(F(R))$ if and only if R is F-consistent. If $(x, y) \in P(R)$ then $(x, y) \in R$ and therefore $(x, y) \in F(R)$. Thus, $(x, y) \in P(F(R))$ for every $(x, y) \in P(R)$ if and only if $(y, x) \notin F(R)$ for every $(x, y) \in P(R)$, or equivalently if and only if $F(R) \cap P^{-1}(R) = \emptyset$. □

The next two results are useful in order to apply Zorn's lemma and show that there is a complete extension of the original preference relation.

Lemma 3. *If $F : \mathcal{R} \to \mathcal{R}$ is closed, monotone and algebraic, then for any countable Z and every chain*

$$R_0 \preceq R_1 \preceq \cdots \preceq R_\alpha \preceq \cdots$$

such that $R_\alpha \in \mathcal{R}_F^Z$ for all α, we have $\cup_{\alpha \geq 0} R_\alpha \in \mathcal{R}_F^Z$.

Proof. Let $B = \cup_{\alpha \geq 0} R_\alpha$. If the chain is finite B is itself an element (the last element) of the chain, so that $B \in \mathcal{R}_F$ is immediate. Thus, we only need to be concerned with infinite chains. We know that each element R_α of the chain is F-consistent (from Lemma 2) and Z-separable, and we only need to show that B is F-consistent and Z-separable.

For consistency of B, assume that there is $(x, y) \in F(B)$ but $(y, x) \in P(B)$. By construction of B we know that $(y, x) \in R_a$ for some relation R_a (with finite index a), and therefore $(y, x) \in R_\alpha$ for $\alpha \geq a$. Since F is algebraic, there is some finite relation $R' \subseteq B$ such that

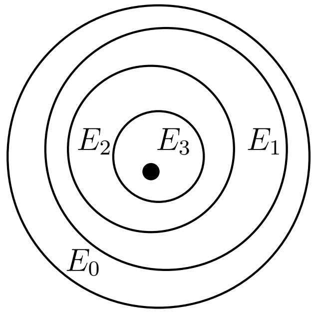


Figure 3: Illustration for Lemma 3. $E_i \subset \mathcal{R}_F^Z$ is the of Z-separable F-consistent extensions of R_i. If every element in the sequence $\{E_i\}$ is non-empty, then the limit relation $\cup_{\alpha \geq 0} R_\alpha$ also has a non-empty set of Z-separable F-consistent extensions.

$(x, y) \in F(R')$. Moreover, since R' is finite, there is some R_b (with finite index b) in the chain such that $R' \subseteq R_b$. Since F is monotone, $F(R') \subseteq F(R_b)$ and therefore $(x, y) \in F(R_b)$. By monotonicity again, $(x, y) \in F(R_\alpha)$ for $\alpha \geq b$. Hence, there is a finite $c = \max\{a, b\}$ such that R_c is not F-consistent, a contradiction.

For Z-separability of B, suppose that $(x, y) \in P(B)$. By construction of B we know that $(x, y) \in R_d$ for some relation R_d (with finite index d), and $(y, x) \notin R_\alpha$ for any α. Hence $(x, y) \in P(R_d)$. From Z-separability of R_d, there is $z \in Z$ such that $(x, z) \in R_d$ and $(z, y) \in R_d$. But then $(x, z) \in B$ and $(z, y) \in B$. □

Lemma 4. *If $F : \mathcal{R} \to \mathcal{R}$ is closed, idempotent, separability preserving, and expansive, then for any countable $Z \supseteq Q_F$ and for every $R \in \mathcal{R}_F^Z$ such that $N(R) \neq \emptyset$ there is a non-empty subset S of $N(R)$ such that $R \cup S \in \mathcal{R}_F^Z$.*

Proof. Consider first the case $R \neq F(R)$, and let $S = F(R) \setminus R$. Note that $S \neq \emptyset$ since F is closed, and by construction $R \cup S = F(R)$. Since F is separability preserving, then $R \in \mathcal{R}_F^Z$ implies $R \cup S$ is Z-separable. Since F is idempotent, $F(F(R)) = F(R)$ so $F(F(R)) \succeq F(R)$ and $F(R) = R \cup S \in \mathcal{R}_F^Z$.

Consider now the case $R = F(R)$. Since F is expansive, there is a nonempty set $S \subseteq N(R)$ such that $R \cup S$ is F-consistent and $P(R \cup S) = P(R)$. Since R is Z-separable, it follows that $R \cup S$ is also Z-separable. Since $R \cup S$ is F-consistent and Z-separable, we get $R \cup S \in \mathcal{R}_F^Z$. □

In order to prove Theorem 1 we need also a classical result from Debreu (1954) included below for reference.

Lemma 5 (Lemma 2 from Debreu (1954))**.** *If R is a complete, transitive and Z-separable preference relation for some countable $Z \subseteq X$, then there is a utility function that represents R and is continuous in any natural topology.*

We turn to the main proof next.

Proof of Theorem 1. Assume throughout the proof that F is a rational closure and R a Z-separable preference relation for some countable Z. Define $Z' = Z \cup Q_F$ and note that R is $Z \cup Q_F$-separable. To prove necessity of the condition in the statement of the theorem, suppose first that R is not F-consistent, and let R' be any extension of R. From $R' \supseteq R$ and monotonicity of F we get $F(R') \supseteq F(R)$. From $P(R') \supseteq P(R)$ we get $P^{-1}(R') \supseteq P^{-1}(R)$. It follows that if R is not F-consistent, then R' is not F-consistent. Note that if R' is not F-consistent, then $P(R') \neq P(F(R'))$ and hence $R' \neq F(R')$. Thus, if R is not F-consistent, it cannot have an extension that is a fixed point of F.

To prove sufficiency, suppose R is F-consistent. Since F is separability-preserving and R is Z'-separable, then $F(R)$ is Z'-separable as well. Let

$$\Omega = \{R' \in \mathcal{R}_F^{Z'} | R \preceq R'\}$$

be the set of extensions of R that are themselves Z'-separable and extended by F. Note that by Lemma 2, $R \in \Omega$ if R is F-consistent, so Ω is nonempty.

We claim that every chain $R_0 \preceq R_1 \preceq \cdots \preceq R_\alpha \preceq \cdots$ of relations in Ω has an upper bound $B = \cup_{\alpha \geq 0} R_\alpha \in \Omega$. To see this, from Lemma 3, $B \in \mathcal{R}_F^{Z'}$. It remains to check that $R \preceq B$. Clearly, $R \subseteq B$. If $P(B) \not\supseteq P(R)$, then there are elements $x, y \in X$ such that $(x, y) \in P(R)$ and $(y, x) \in B$. But then there must be a relation R_α in the chain for which $(y, x) \in R_\alpha$, which contradicts the fact that $R \preceq R_\alpha$ and we conclude that $B \in \Omega$. Clearly, $\preceq$ is a partial order (reflexive, antisymmetric and transitive binary relation) on Ω and we just showed that every chain has an upper bound. Hence, by Zorn's lemma, there is maximal element of Ω, and we can denote it by R^*.

We claim that R^* is complete. To see this, assume on the contrary that $N(R^*) \neq \emptyset$. From Lemma 4 and the fact that $Q_F \subseteq Z'$, we know that there is a nonempty $S \subseteq N(R^*)$ such that $R^* \cup S \in \mathcal{R}_F^{Z'}$. Since $R^* \succeq R$, we have $R^* \cup S \supset R^* \supset R$. Note that $(y, x) \in P(R)$ implies $(x, y) \notin R^*$ (since $R^* \succeq R$) and $(x, y) \notin S$ (since $S \subseteq N(R^*) \subseteq N(R)$). Hence, $P(R^* \cup S) \supseteq P(R)$. But then $R^* \cup S \succeq R$ and $R^* \cup S \in \mathcal{R}_F^{Z'}$, which contradicts that R^* is a maximal element of Ω.

We claim further that R^* is a fixed point of $F(R)$, i.e. $F(R^*) = R^*$. To see this, note that $R^* \subseteq F(R^*)$ follows from the fact that $R^* \preceq F(R^*)$. To get the reverse, assume that $(x, y) \in F(R^*)$ and $(x, y) \notin R^*$. From completeness of R^*, (y, x) must be an element of $P(R^*)$ which contradicts $R^* \preceq F(R^*)$. Therefore, $F(R^*) \subseteq R^*$.

We are left to show that there is a utility function that represents $R^* = F(R^*)$. We just showed that R^* is complete. Since F is a rational closure, R^* is transitive as well. As we already showed $R^* \in \Omega \subseteq \mathcal{R}_F^{Z'}$, it follows that R^* is Z'-separable. Hence, R^* satisfies the conditions from Lemma 5, i.e. there is utility function that represents R^*. Moreover, the utility function is continuous in natural topology. □

Intuitively, we can think of the rational closure F as helping to construct a complete extension of the original preference relation via an iterated algorithm. Starting with the

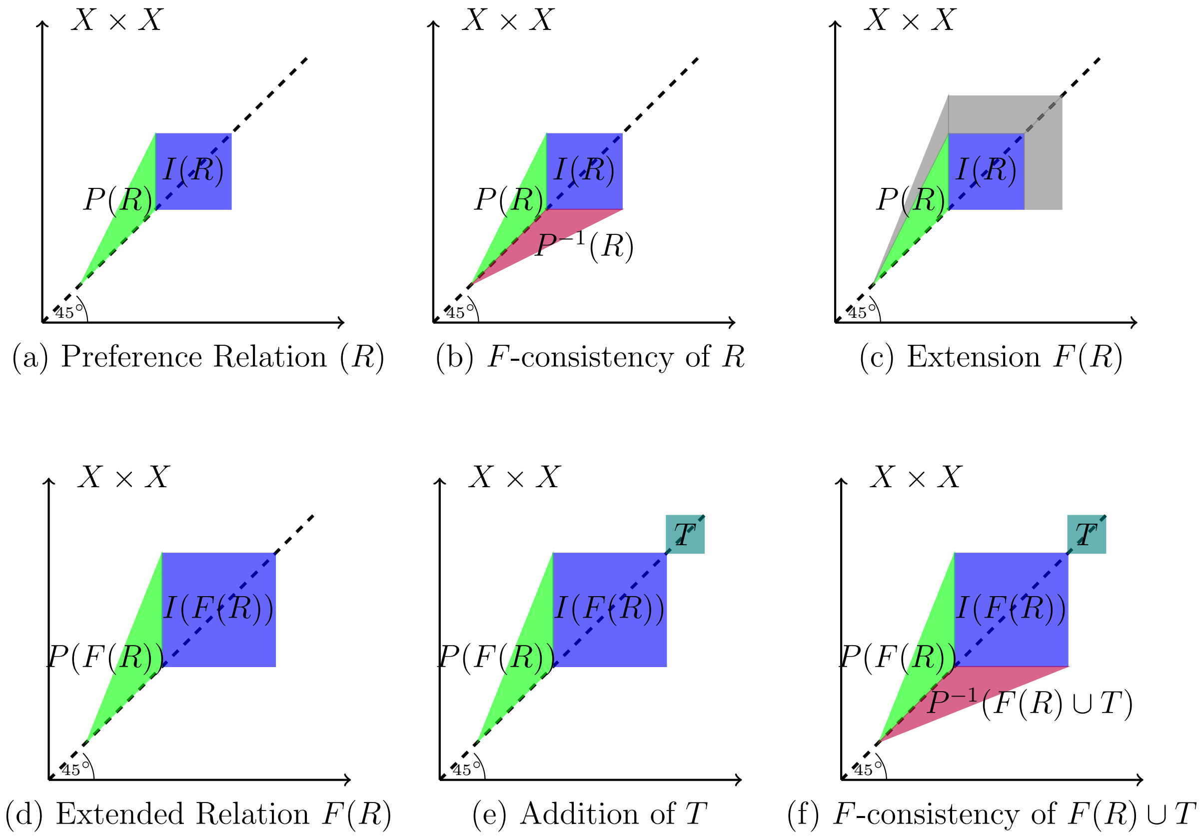


Figure 4: Intuition for the proof of Theorem 1. The dashed line is the diagonal. If (x, y) lies above the diagonal, then (y, x) is the symmetric point below the diagonal.

original preference relation, the algorithm works as follows. If the preference relation at a given iteration is already a fixed point of F, but it is not a complete relation, the algorithm requires adding indifference pairs to R while keeping the new preference relation in $\mathcal{R}_F$. Adding indifference points in this manner is possible since F is expansive. If instead the preference relation at a given iteration is not a fixed point, the algorithm requires going from R to $F(R)$, which expands R while preserving its asymmetric part—since F is idempotent, the new preference relation is a fixed point of F. In this sense, the proof of Theorem 1 establishes that such algorithm converges to a complete fixed point as long as the original preference relation is F-consistent. Since the algorithm preserves separability properties of the preference relation, and fixed points of F are transitive, the complete fixed point is representable by a utility function. Of course, convergence need not occur after a finite number of iterations, and the proof relies on Zorn's Lemma to assert existence.

4 Revealed Preference Revisited

In this section we illustrate the techniques proposed in the paper by revisiting the classical problem of the existence of a utility function rationalizing observations obtained from a finite number of budget sets. Formally, a **consumption experiment** is a finite vector $E = (x_i, B_i)_{i=1}^n \in (X \times 2^X)^n$ where for each $i = 1, \dots, n$, $x_i \in B_i \subseteq X$. The interpretation is that x_i are chosen alternatives and B_i are budget sets, so that each x_i is (directly) revealed to be strictly preferred to each alternative in $B_i \setminus \{x_i\}$.

Given a consumption experiment $E = (x_i, B_i)_{i=1}^n$, for each $i = 1, \dots, I$, let $R_i = \{(x_i, y) : y \in B_i \setminus \{x_i\}\}$, and let $R_E = \bigcup_i R_i$. We say that an experiment $E = (x_i, B_i)_{i=1}^n$ can be **rationalized** if there is a preference relation that is a complete extension of every element in the set $\{R_i\}$ and that can be represented by a utility function. We claim:

Proposition 1. *A consumption experiment can be rationalized if and only if (1) $R_E \succeq R_i$ for $i \in \{1, \dots, n\}$ and (2) R_E is T-consistent.*

Proof. To prove sufficiency of conditions 1 and 2, note that R_E is separable with respect to the finite set $\{x_1, \dots, x_n\}$ and recall that the transitive closure is a rational closure (Lemma 1). From Theorem 1, then, R_E has a complete extension R^* that can be represented by a utility function if and only if R_E is T-consistent; that is, condition 2. Since $R^* \succeq R_E$ and $\succeq$ is a transitive relation, condition 1 is sufficient for $R^* \succeq R_i$ for $i \in \{1, \dots, n\}$.

To prove necessity of condition 1, note that by construction $P(R_i) = R_i$ for $i \in \{1, \dots, n\}$. Hence, if there is some i such that $P(R_E) \not\supseteq P(R_i)$, it must be the case that there is some $j \neq i$ such that $(x_i, x_j) \in R_i = P(R_i)$ and $(x_j, x_i) \in R_j = P(R_j)$. But there cannot be any preference relation R^* satisfying $(x_i, x_j) \in P(R^*)$ and $(x_j, x_i) \in P(R^*)$.

To prove necessity of condition 2, suppose R_E is not T-consistent but condition 1 holds. Then for any preference relation R^* that extends every R_i, we can build a cycle of strict preference between three or more alternatives, which implies that R^* cannot be represented by a utility function. □

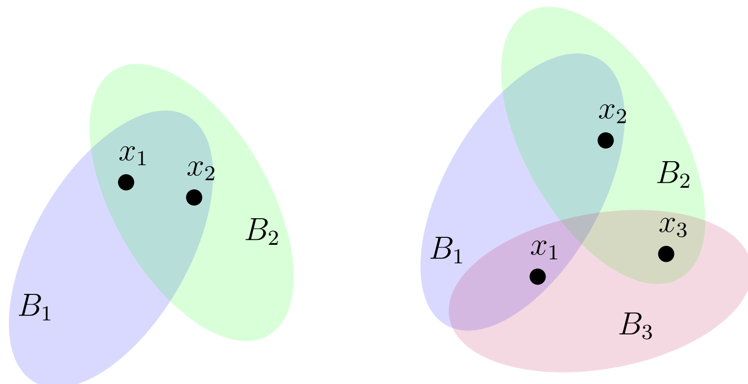
Two criteria to ascertain the rationality of the consumption experiment (developed first by Samuelson (1938) and Houthakker (1950)) are described below:

Definition 8. *The consumption experiment $E = (x_i, B_i)_{i=1}^n$ satisfies the Weak Axiom of Revealed Preference (**WARP**) if for every $\{i, j\} \subseteq \{1, \dots, n\}$, $x_j \in B^i$ implies $x_i = x_j$ or $x_i \notin B_j$.*

Definition 9. *The consumption experiment $E = (x_i, B_i)_{i=1}^n$ satisfies the Strong Axiom of Revealed Preference (**SARP**) if for every integer $m \leq n$ and every $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$, $x_{i_{j+1}} \in B_{i_j}$ for $j = 1, \dots, m-1$ implies $x_{i_1} = x_{i_m}$ or $x_{i_1} \notin B_{i_m}$.*

The interpretation of WARP is that two alternatives cannot be directly revealed to be strictly preferred to each other, while the interpretation of SARP that in addition two alternatives cannot be indirectly revealed to be strictly preferred to each other, via a chain of direct revelation. (See Figure 5.)

The following are immediate:



Violation of WARP   Violation of SARP, but not WARP

Figure 5: Relation between WARP and SARP

Lemma 6. *$E = (x_i, B_i)_{i=1}^n$ satisfies WARP if and only if $R_E \succeq R_i$ for $i \in \{1, \ldots, n\}$.*

Lemma 7. *$E = (x_i, B_i)_{i=1}^n$ satisfies SARP if and only if (1) $R_E \succeq R_i$ for $i \in \{1, \ldots, n\}$ and (2) R_E is T-consistent.*

As a corollary of Proposition 1 and Lemma 7, a finite consumption experiment can be rationalized if and only if it satisfies SARP.

By working with other closures we can induce monotonicity as well as transitivity in the complete extension of the original preference relation. For this purpose we need to introduce more structure on X. Assume X is endowed with a transitive and reflexive relation $\geq$, with strict part denoted by $>$, and suppose there is a countable set Q that is dense in X with respect to $\geq$; that is, for all $x, y \in X$ such that $x > y$ there is $Z \in Q$ such that $x > z > y$. As an example, we have $X = \mathbb{R}^m$ and $Q = \mathbb{Q}^m$ for positive integer m. We say that a preference relation R is **monotone** if for all $x, y \in X$, $x > y$ implies $(x, y) \in P(R)$.

We define the **monotone closure** by $M : \mathcal{R} \to \mathcal{R}$, where $(x, y) \in M(R)$ if there is a finite sequence $s_1, \ldots, s_n$ such that $s_1 = x$ and $s_n = y$, and for any $j = 1, \ldots, n-1$ either (1) $(s_j, s_{j+1}) \in R$, or (2) $s_j > s_{j+1}$.

Lemma 8. *The monotone closure $M : \mathcal{R} \to \mathcal{R}$ is a rational closure.*

Proof. It is easy to check that M is an algebraic closure and that induces transitivity. Expansiveness of M can be proven in a similar way to the expansiveness of T, considering the fact that $R = M(R)$ is already monotone relation, i.e. all pairs $x \geq y$ are already in $M(R)$.

We are left to show that M is separability-preserving. Consider a preference relation R satisfying $R \in \mathcal{R}_M^Z$ for some countable set $Z \subseteq X$. We claim that $M(R)$ is $Z \cup Q$-separable, so that $Q_M = Q$. To see this, note that $(x, y) \in P(M(R))$ implies that there is a sequence $S = s_1, \ldots, s_n$, such that $s_1 = x$, $s_n = y$ and for any $j = 1, \ldots, n-1$ either $(s_j, s_{j+1}) \in R$, or $s_j > s_{j+1}$, with at least one $k \in \{1, \ldots, n-1\}$ such that either (1) $(s_k, s_{k+1}) \in P(R)$, or (2) $s_k > s_{k+1}$. If $(s_k, s_{k+1}) \in P(R)$, then there is $z \in Z$, such that $\{(x, z), (z, y)\} \subseteq M(R)$.

If $s_k > s_{k+1}$ there is $z \in Q$ (since Q is dense with respect to $\geq$) such that $s_k \geq z \geq s_{k+1}$, i.e. $\{(x, z), (z, y)\} \subseteq M(R)$. □

We make the (relatively mild) assumption that budgets are **comprehensive**; that is for each $i \in \{1, \ldots, n\}$, $x \in B_i$ and $y < x$ imply $y \in B_i$.

We have:

Proposition 2. *A consumption experiment with comprehensive budgets can be rationalized by a strictly increasing utility function if and only if (1) $R_E \succeq R_i$ for $i \in \{1, \ldots, n\}$ and (2) R_E is M-consistent.*

Proof. We prove sufficiency of conditions (1) and (2); necessity of each of the two conditions follows along the lines of the previous proposition.

From condition (2) and Lemma 2, we have that $M(R_E)$ extends R_E. From condition 1, then, $M(R_E)$ extends R_i for $i \in \{1, \ldots, n\}$. Since $(x_i, y) \in P(R_i)$ for all $y \in B_i \setminus \{x_i\}$, we must have $(x_i, y) \in P(M(R_E))$ for all $y \in B_i \setminus \{x_i\}$. But from the definition of M, $y > x_i$ implies $(y, x_i) \in M(R_E)$. It follows that $(x, y) \in R_i$ implies that it is not the case that $y > x$, and hence $(x, y) \in R_E$ implies that it is not the case that $x < y$.

We claim that $M(R_E)$ is monotone; that is $(x, y) \in P(M(R_E))$ for all $x > y$. To see this, from the definition of M, $x > y$ implies $(x, y) \in M(R_E)$. So we only need to show $x < y$ implies $(x, y) \notin M(R_E)$, or equivalently, $(x, y) \in M(R_E)$ implies that it is not the case that $x < y$. That is, it remains to be shown that if there there is a sequence $s_1, \ldots, s_n$ such that $s_1 = x$ and $s_n = y$, and for any $j = 1, \ldots, n-1$ either (i) $(s_j, s_{j+1}) \in R_E$, or (ii) $s_j > s_{j+1}$, then it cannot be the case that $x < y$.

Consider any such sequence as described in the previous paragraph. Trivially, if every consecutive pair in the sequence is of type (ii) we get $x > y$, so assume there is some consecutive pair of type (i) in the sequence, and let (s_k, s_{k+1}) be the last step of type (i). From the definition of E, this implies that s_k is equal to x_i for some $i \in \{1, \ldots, n\}$. If $k + 1 = n$, we get immediately $(s_k, y) \in R_i$. If $k + 1 < n$, using the fact that (s_k, s_{k+1}) is the last step of type (i) we get $y < s_{k+1}$, hence from comprehensiveness of budget sets $y \in B_i$. Since $y < s_{k+1} \in B_i$, we know $y \neq x_i$ and then $(s_k, y) \in R_i$. In either case, then, from condition (1), $(s_k, y) \in P(R_E)$. But if $y > x$, we can show $(y, s_j) \in M(R_E)$ using the sequence $y, x, s_1, \ldots, s_j$, which violates condition (2).

Note that $M(R_E)$ is (trivially) M-consistent, and it is separable with respect to the countable set $\{x^1, \ldots, x^\tau\} \cup Q$. Since, from Lemma 8, M is a rational closure, it follows from Theorem 1 that $M(R_E)$ has a complete extension $R^* = M(R^*)$ that can be represented by a utility function. Since $M(R_E)$ is monotone and R^* is an extension of $M(R_E)$, it follows that R^* is monotone. This in turn implies that any utility function representing R^* is strictly increasing.

Since R^* is an extension of $M(R_E)$, it follows from conditions (1) and (2) and Lemma 2 that it is an extension of R_i for $i \in \{1, \ldots, n\}$ as well. □

Note that consistency with the monotone closure implies that for every budget B_i there is no point above the chosen alternative x_i; we do not need budgets to be comprehensive to establish this. This implies that directly observed preferences do not contradict monotonicity. Comprehensive budgets help us in proving that consistency with the monotone closure implies that preferences built using the closure do not contradict monotonicity either.

5 Generalized Revealed Preference Revisited

Varian (1982) introduces an approach to revealed preference in which observed choices are revealed to be strictly preferred to alternatives that are in the budget set and that are cheaper than other alternatives in the budget set, and observed choices are revealed to be weakly preferred to alternatives that are in the budget set but are not cheaper than other alternatives. Intuitively, observed choices are possibly indifferent to other alternatives in the budget set.

To formalize this approach in our environment, we assume as in the previous section that X is endowed with a transitive and reflexive relation $\geq$, with strict part denoted by $>$, and suppose there is a countable set Q that is dense in X with respect to $\geq$; that is, for all $x, y \in X$ such that $x > y$ there is $Z \in Q$ such that $x > z > y$.

Given a consumption experiment $E = (x_i, B_i)_{i=1}^n$, for each $i = 1, \ldots, I$, let R_i and R_E be defined as in the previous section, and let $\tilde{R}_i = \{(x_i, y) : y \in B_i \backslash \{x_i\}$ and $y < x$ for some $x \in B_i\}$ and let $\tilde{R}_E = \bigcup_i \tilde{R}_i$.

We say that an experiment $E = (x_i, B_i)_{i=1}^n$ can be **rationalized with possibly indifferent choices** if there is a monotone preference relation that is a complete extension of every element in the set $\{\tilde{R}_i\}$ and of $T(R_E)$ and that can be represented by a utility function.

We have:

Proposition 3. *A consumption experiment with comprehensive budgets can be rationalized with possibly indifferent choices by a strictly increasing utility function if and only if (1') $T(R_E) \succeq \tilde{R}_i$ for $i \in \{1, \ldots, n\}$ and (2') $T(R_E)$ is M-consistent.*

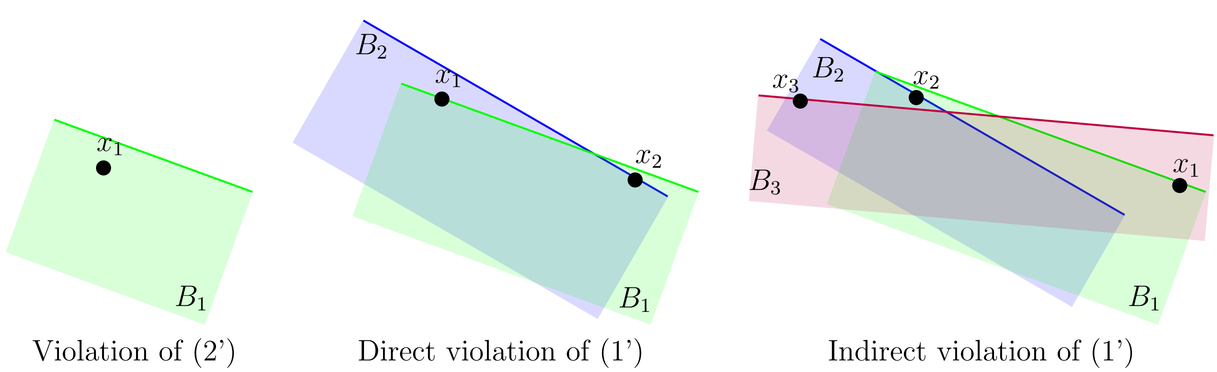


Figure 6: Relation between (1') and (2')

The proof is analogous to the proof of Proposition 2.

Adapting the formulation by Varian (1982) to our setting, we can define

Definition 10. *The consumption experiment* $E = (x_i, B_i)_{i=1}^n$ *satisfies the Generalized Axiom of Revealed Preference (**GARP**) if* $(x_i, x_j) \in T(R_E)$ *implies that there is no* y *such that* $y \in B_j$ *and* $y > x_i$.

With comprehensive budgets, GARP can be restated as $(x_i, x_j) \in T(R_E) \Rightarrow (x_j, x_i) \notin \tilde{R}_j$. We claim that Conditions (1') and (2') in Theorem 3 are jointly equivalent to GARP plus the assumption that observed choices are maximal with each budget set, i.e. for any $i \in \{1, \ldots, n\}$ there is no $y \in B_i$, such that $y > x_i$. (See Figure 6.) Maximality of observed choices is assumed in the original paper by Varian (1982).

Lemma 9. *The consumption experiment* $E = (x_i, B_i)_{i=1}^n$ *with comprehensive budgets satisfies GARP and maximality of observed choices if and only if (1')* $T(R_E) \succeq \tilde{R}_i$ *for* $i \in \{1, \ldots, n\}$ *and (2')* $T(R_E)$ *is* M*-consistent.*

Proof. To prove sufficiency, assume first that there is a violation of GARP, that is there is $(x_i, x_j) \in T(R_E)$ and $(x_j, x_i) \in \tilde{R}_j$. Therefore, $(x_j, x_i) \in P(\tilde{R}_j)$ and $(x_i, x_j) \in T(R_E) \cap P^{-1}(\tilde{R}_j)$. Hence, $\tilde{R}_j \not\preceq T(R_E)$, i.e. a violation of (1'). Assume instead there is a violation of maximality of observed choices, i.e. there is $y > x_i$ and $y \in B_i$. Then $(y, x_i) \in M(T(R_E))$ and $(x_i, y) \in P(T(R_E))$. Hence $T(R_E)$ is not M-consistent, i.e. there is a violation of (2').

To prove necessity of condition (1'), suppose it is violated. We have then that there is $(x_i, x_j) \in \tilde{R}_i$ and $(x_j, x_i) \in T(R_E)$. And $(x_i, x_j) \in \tilde{R}_i$ implies that there is $y > x_j$ such that $y \in B_i$; which violates GARP.

To prove necessity of condition (2'), suppose it is violated. Then there is $(x_i, y) \in P(T(R_E))$ and $(y, x_i) \in M(T(R_E))$. Since $T(R_E)$ is transitive and $>$ is transitive as well we can claim that $y > x_i$. At the same time $(x_i, y) \in T(R_E)$ implies that there is j, such that $y \in B_j$ and $x_j \in B_i$, hence $(x_i, x_j) \in T(R_E)$. Recall that budgets are comprehensive, therefore, $x_i \in B_j$, because $y > x_i$. Then $(x_j, x_i) \in \tilde{R}_j$; which violates GARP. □

As a corollary of Proposition 3 and Lemma 9, an extended version of GARP (including maximality of observed choices) is necessary and sufficient for rationalization with possibly indifferent choices by a strictly increasing utility function.

	$\emptyset$	T-consistency	M-consistency
$\{R_i \preceq R_E\}$	WARP	SARP	$*$
$\{\tilde{R}_i \preceq T(R_E)\}$	$**$	$***$	GARP

Table 1: A cheat sheet of consistency conditions and revealed preference axioms

Table 1 above summarizes the relation between revealed preferences axioms and the conditions on extensions and closures we use to obtain representations. We think of row conditions as criteria regarding the internal consistency of the observed choices, i.e. the

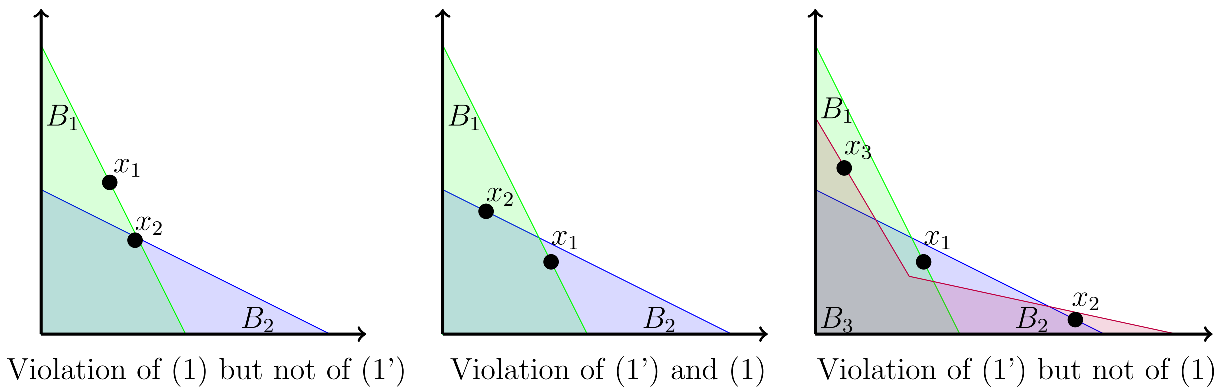


Figure 7: Relation between internal consistency conditions

consistency of each of the observed choices with the complete dataset. By the same token, we think of column conditions as criteria regarding the external consistency of the dataset, i.e. its consistency with theories about how a complete preference ordering should look like. External consistency conditions grow more demanding as we move from left to right. Internal consistency conditions cannot be similarly ranked; while $P(\tilde{R}_i) \subseteq P(\tilde{R}_i)$, $P(R_E)$ is not necessarily a subset of $P(T(R_E))$. (See Figure 7.)

The unnamed cells in Table 1 are of some interest. $(*)$ gives us necessary and sufficient conditions for representation by an increasing utility function in Proposition 2, and thus is just a version of SARP. $(**)$ is the Weak version of GARP, requiring that observations do not directly contradict rationalization by monotone preferences allowing for indifference of observed choices. $(***)$ gives us necessary and sufficient conditions for representation by a utility function assuming all observations come from a region in the consumption space where preferences are monotone.

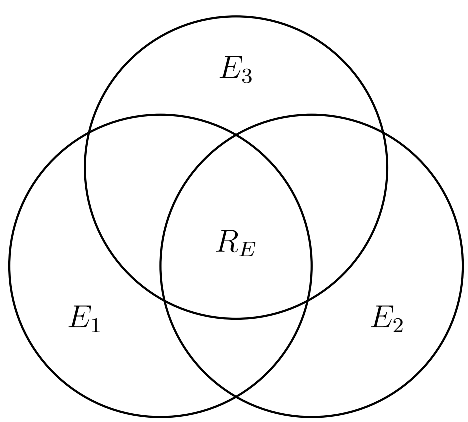
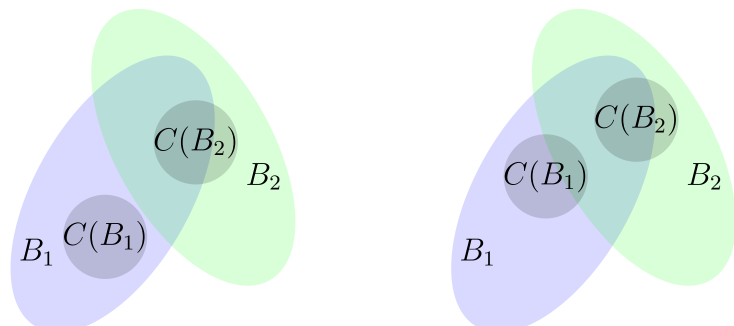


Figure 8: Consistency conditions. E_i is the set of F-consistent Z-separable extensions of R_i. Internal consistency requires the aggregated preference relation (R_E) to be a F-consistent Z-separable extension of R_i for all i.

Our approach allows to restate the revealed preference axioms in a very abstract and general form. Neither internal nor external consistency conditions require unique observed choices from budget sets. The internal consistency condition is built on the idea that there are some elementary observed relations (R_i and $\tilde{R}_i$ in the examples above), and that the aggregated revealed preference relation (R_E and $T(R_E)$ in the examples above) must be an extension of elementary ones. The external consistency condition, in turn, is built on the idea that the aggregated observed preference relation can be extended to a complete preference relation satisfying properties like transitivity and monotonicity. This framework allows is to consider, from the perspective of revealed preferences, observed behavior in environments beyond the classical one of the a sequence of single alternatives chosen from budgets sets. Observed behavior may be set-valued, for instance, in situations in which there are many opportunities to choose from the same budget (see Figure 9). And choice sets may be very different from linear budget sets (e.g. multiple price lists in experimental economics).



Internally consistent demand   Internally inconsistent demand

Figure 9: Internal consistency of set-valued demands

6 Conclusion

In this paper we show that there is a complete extension of an incomplete preference relation that is fixed point of a mapping over preferences (the rational closure) and can be represented by a utility function if and only if the original preference relation satisfies a congruence condition related to the specific mapping. Intuitively, a rational closure is a rule that can be used to extend an incomplete preference relation. The proof of the theorem relies on the alternated application of the rational closure and the addition of indifference pairs to construct the extension.

An advantage of using an explicit rule to construct the complete extension of the original preference relation is that further desiderata on the utility function can be induced by the rule. We illustrate this point by revisiting the classical revealed preference problem of the existence of a rationalization of a sequence of observed choices by means of strictly increasing utility functions. We show, in particular, that classical revealed preference axioms can be recast in terms of extensions and closures in a very abstract way, encompassing situations that do not reduce to a sequence of single choices from linear budget sets.

Rational closures as defined in this paper construct preferences over each ordered pair of alternatives not in the original preference relation employing only a finite number of observations regarding other pairs of alternatives. This "algebraic" requirement is necessary for the usage of Zorn's lemma in the proof of existence of the complete extension. However, this requirement is not compatible with a rule that induces continuity in order to construct the complete extension. Thus, constructing a continuous extension of the original preference relation seems an elusive goal in the general case,[7] i.e. without assumptions such as Euclidean consumption spaces and further constraints over budget sets.

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[7]That is, for a fixed topology; continuity of the complete extension holds by assumption for natural topologies.

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