Stonemasons, Artists, and Rational Numbers

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Theoretical Framework

Differences between the mathematics content knowledge possessed by Chinese elementary school teachers and their counterparts in the United States as well as differences in these teachers' beliefs about the teaching and learning of mathematics have been found by the research of Correa, Perry, Sims, Miller & Fang (2008). They discovered that Chinese teachers stress the applications of mathematics to the children's lives and capitalized on the children's mathematical interest. Teachers in the United States, however, appear to be focused on a variety of learning styles, individualizing learning plans, and, perhaps, a dependence on formulaic problem solution strategies.

According to the National Council of Teachers of Mathematics (NCTM) (2009), there has never been a greater need for understanding of and ability to use mathematics in life and in work; and the need continues to grow. And, it is useful to note that mathematics is present in everyone's daily routines. The lives of children are full of exposure to the idea of numbers and probability. And, the NCTM recommends that teachers capitalize on this exposure by connecting school math with students' home experiences. The most meaningful connections are made when lessons incorporate students' relevant personal experiences in math lessons, not to situations contrived by others (Ensign, 1997). This is not a new idea. In promoting child-centered education, Dewey (1938) stated that educational material which is developed and presented in ways which are unrelated to the child's world or the child's experiences are of dubious value.

Rational numbers tend to be one of the most difficult concepts for students to learn during elementary school. (At this point, it is useful to acknowledge the difference between the terms *rational number* and *fraction*. Although they are frequently used interchangeably, they do not have the same meaning. Any rational number can be written in fractional form; however, merely taking on fractional form does not ensure a rational number, e. g., $\frac{\pi}{2}, \frac{\sqrt{3}}{4}, \frac{6}{i}$. However, for purposes of this discussion, no irrational or non-real numbers are involved and the terms *rational number* and *fraction* will be regarded as having the same meaning.) Clearly, a solid understanding of rational numbers is requisite for moving on to more complex problem solving in the intermediate and higher grades; else, as students progress from one math course to another, some can get deeply, profoundly lost. Although it is possible to pick up virtually any chapter in a history book and learn something new, an advanced chapter of math could be totally incomprehensible to a student who is unfamiliar with, uncomfortable with, or fearful of the concepts. Because the risk of frustration is high, there is a need for math teachers who can ably explain complex topics (Chandler, 2009). Consequently, there is a need for teachers who understand the concepts so deeply that their conceptual knowledge is impacted. There is no shortage of research which reveals that there are teachers who do not have a firm grasp on fractional concepts. Noting the difficulties of both learning and teaching mathematics topics which involve multiplicative structures, Lamon (2007) suggests:

> Of all the topics in the school curriculum, fractions, ratios and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in high mathematics and science, and one of the most compelling research sites.

Incorrect procedures may hold some logic for the child, as was illustrated so extremely by Erlwanger's (1973) Benny, who developed a system for converting his answers to the ones he found available on the answer sheets. The process of manipulating fractions may have some "logic" for the child, even though that logic is flawed. Burns (1997) asserts that the errors which students make are not a random mishmash but rather remarkably consistent; most prevalently, the children's' errors are rule-bound, the result of applying an incorrect procedure in place of a correct algorithm. Often the child has been taught the rule by a teacher but applies the rule in an inappropriate situation. Constructivists use the Benny case to illustrate what dire consequences can result when mathematical thinking is not an element of the curriculum (Noddings, 1995). On the basis of what we have learned in recent years about human learning, i. e., it works best when it is comprised of constructing meanings rather than receiving them, such an approach results in better teaching of mathematics than non-participatory, non-communal, more traditional methods (Bruner, 1996)

However, many teachers' mathematics knowledge is generally problematic in terms of what they know and how they process their knowledge of mathematics concepts or processes, including fundamental concepts from school mathematics curriculum. They do not always possess a deep, broad, and thorough understanding of the content they are to teach (Da Ponte & Chapman, 2003). Our traditional teaching for computational ability, Lamon (2007) contends, had left us pedagogically bankrupt for an age that values connections and meaning, and basic questions needed exploration. This means that students need to understand logical relationships between mathematical terms and need to be able to see structural connections behind the algorithms (Skovmose, 2005). If we can accomplish this, the students would abandon the rote plug-and-play scenario of implementing a formula and rise to levels of comprehension which would enable them to understand why a certain formula or procedure should be used and how to use it.

In the 1980s, according to Da Ponte & Chapman (2003), there were a low number of academic papers which addressed teachers' knowledge of mathematics teaching. In one of those papers, Andelfinger (1981) presented his findings that the teachers in his survey regarded fractions and decimals as separate topics and that they viewed the two topics as having neither problems nor difficulties in common as well as having little relationship to other topics. While this lack of understanding about the correlation between fractions and decimals by teachers, only thirty years ago, is at least surprising, if not disturbing, Sowder, Philipp, Armstrong, & Schappelle (1998) had similar findings almost twenty years later. Their research discovered that preservice and novice teachers confused division of fractions with multiplication of fractions.

For example, findings from investigations by Phillpou and Christou (1994) into the conceptual and procedural knowledge of fractions of preservice elementary school teachers indicated that these teachers possessed only a limited understanding of the ideas underlying the conceptual knowledge of fractions. In another study published that same year, Zazkis and Campbell (1994) researched preservice elementary teachers' understanding of concepts and found strong dependence upon procedures. The researchers concluded that these procedural attachments compromised and thwarted the teachers' ability to develop more refined and more meaningful structures of conceptual understanding. After all, if the teacher does not have a firm conceptual understanding of the meaning of fractions and the applications of fractions, that teacher's ability to create meaningful lessons and to demonstrate applications to real-world examples which are relevant to the children's lives is severely hampered.

In sharp contrast, Ma (1999) recalls that she learned division by fractions through her teacher's focus on the relationship between division by fractions and division by positive integers. She notes "division remains the inverse of multiplication, but meanings of division by fractions extend meanings of whole numbers divisions, e.g., the measurement model for finding out how many halves are in $1\frac{3}{4}$ and the partitive model for finding a number such that half of it is $1\frac{3}{4}$." She credits her own teacher's deep understanding of rational numbers with the ability to teach effectively the subject and to lead students to discover appropriate connections. Ma (1999) also describes one U.S. teacher who, unlike her colleagues who used manipulatives for computational illustration, used manipulatives to demonstrate mathematical concepts. Ma (1999) asserts that this teacher's deep understanding of mathematical topics was the sole reason behind her ability lead her students to mathematical thinking beyond the procedural realm.

Research Questions

Three ideas are abundantly clear: teachers cannot teach what they do not know; teachers generally do not have deep understanding of rational numbers; and, student learning is directly related to their teachers' conceptual understanding. So, the concern is ensuring that teachers possess the conceptual understanding which they need in order to teach their students effectively and to incorporate relevant examples in their lessons? The research questions are:

1. How do mathematics teachers develop deep understanding of rational numbers?

2. How can professional development for K-12 mathematics teachers be designed and implemented so that the teachers develop deep understanding of rational numbers?

Context of the Study

The I.M.P.A.C.T. (Improving Mathematical Practices via Algebraic Connections & Technology) summer institute is the innovation of two George Mason University professors, Dr. Jennifer Suh and Dr. Padhu Seshaiyer. The purpose of the project is the development of mathematical teaching knowledge through a collaborative network of pre-service and in-service teachers who "collaboratively plan lessons and exchange best instructional practices and effective uses of "tech-knowledgy" tools to design instructional tasks which promote algebraic conceptual thinking. Teacher collaboration enhances their professional practice which then affects students' learning." Their idea came to fruition during the first week of August 2010, when 85 Northern Virginia K-8 public school teachers from four counties arrived at the George Mason University Fairfax campus for the kick-off meeting. (The participants in the institute will be referred to as *teachers*. The four instructors will be referred to as *instructors*.)

For the week's activities, teachers were grouped by county and wore color-coded nametags. Four large, adjacent conference rooms were utilized. A large meeting area was centrally located. Teachers were assigned to a homeroom (Room B, C, D, or E). Each room was furnished with four long rectangular tables at which six to eight teachers were seated. For the morning activities, they reported to the same room each day while the instructors moved to a different room each day. In this way, each group saw each of the four instructors, but on different days. Additionally, teachers were also randomly assigned to afternoon groups. These groups attended workshops and had the opportunity to interact with teachers from different counties.

Teachers had reading and homework assignments every evening. Starting on Tuesday, the first hour of each day was dedicated to discussion of the homework assignment. Then, the instructor presented an opening problem for which the class had a limited time to develop a solution strategy. Each of these opening problems was used in every classroom on a given day; that is, on Tuesday, all four rooms had the same opening problem. On Wednesday, all four rooms had the same opening problem, etc.

After the opening problem was completed and discussed, the instructor would move to the first daily topical problem and direct the class to find a solution without using any formulas or rote procedures, such as cross-multiplication. The idea was the development of conceptual knowledge, not procedural skill. Each group, one or two groups at each table, discussed the problem, and recorded their thought processes on large (20" x 23") Post-it® poster paper (see Appendix B). Teams affixed their posters to the wall and then explained their reasoning, their discussions, their mistakes, and their conclusions with the class. Others in the class could comment or ask questions. After the mid-morning break, the instructor posed a second daily topical question and the process began again. For some questions, groups were required to strategize solutions by at least three of the five possible representations. When discussions were rich, instructors were hesitant to halt them in favor of strictly maintaining a time line. When time was short, the teams would affix their posters to the wall, but, in lieu of verbal explanations from the teams, the class would do a "gallery walk," a walk around the room stopping to look at and to analyze each poster. The instructor ensured that the groups moved from poster to poster in a timely fashion so that everyone had a chance to see every poster and so that no bottlenecks were formed.

The daily topics included reasoning up and down, direct and inverse thinking, unitizing, and, ratios and proportional thinking. Afternoon sessions included classroom technology, writing assessments, and lesson study.

The teachers wrote reflections each day on the daily concept, using one of the classroom problems. A comprehensive course reflection is due at a future date. The purposes of the daily reflections are to have the teachers: describe a rational number problem and explain their thoughts and solution strategies; identify any differences in their own understanding, approaches, and thinking which resulted from the day's activities; and, illuminate any modifications to their teaching content and approach which they intend to employ.

Participants were dismissed at approximately 3:30 p.m. each day. Afterward, Dr. Suh, Dr. Seshaiyer, the four instructors, and I met for approximately one hour to discuss the day and strategize for the next day. Discussions centered around how the teachers were grasping the material, what difficulties were encountered, what, if any, changes should be made, the accuracy of their time predictions, and general observations. The opening problem, which, again, was the same for all four groups each morning, was discussed and a choice was made. Alternate problems for each class were also discussed, in the unexpected event that a class had extra time available. There was never any need for these; two investigative questions took all the available time.

Teachers have several assignments to complete in the coming months. Due dates for all assignments have been established and are publicized. All teachers have access to the course website and can submit their work electronically. All participants will meet again in March 2011 to wrap up the course.

Participating teachers will receive three graduate credits for completing the course satisfactorily and each will receive a small stipend, as allowed by their respective counties.

Method

Each day, I attended morning sessions with a different instructor. I observed the class participants, listened to their discussions, and studied their posters. I also took photographs and did some video recording. Each day, I discussed my observations with Dr. Suh, Dr. Seshaiyer, and the four instructors. I was also fortunate to have the unexpected opportunity to fill in briefly as an instructor twice during the week, when the scheduled instructors were unavailable. One instructor had an inflexible commitment on Tuesday morning and another instructor had one of Friday morning.

For this study, I decided to focus on one problem and I chose the cathedral problem (see Appendix A). This was the first problem which I observed a class discussing. I also led the discussion on this question the next morning. I heard rich discussions and witnessed discovery during the solving of this problem. Several of the teachers made comments to me which I consider valuable.

Photographs of all of the posters of the cathedral problem were taken and arranged in order of the day of completion. Room E completed this problem first, followed by Rooms B, C, and D, respectively. The data from the posters were analyzed for content, connections between concepts, and any possible differences related to the time already spent in the seminar.

I also read the reflections from the teachers which related to the cathedral problem and looked for common themes as well as individual perspectives.

Data analysis

My analysis of the poster artifacts will be presented first, followed by my analysis of the teacher reflections.

Room E was the first homeroom to work on this problem. Their artifacts are shown in Figures 1-5 in Appendix B. The first group, Figure 1, argued that if one artist and one stonemason together made 11 dollars, then the total for three of each would be 33 dollars. However, they reasoned that because we know that 33 dollars is enough to pay those six workers plus another stonemason, then one stonemason and one artist must together make less than 11 dollars. This group then used the guess-and-check method. They first assumed that the total for one artist and one stonemason was 8 dollars. They tried the combination of 1 dollar for the cost of one artist and 7 dollars for the cost of one stonemason (8 dollars total); however, they discovered that the total for 3 artists and 4 stonemasons was less than the needed 33 dollars. They tried other combinations but saw that the total cost was decreased; so, they abandoned the idea of an 8-dollar total. They then assumed that the total for one artist and one stonemason was 9 dollars. However, their starting guess for the cost of one artist was 2 dollars, not 1 dollar. They found that the combination of three artists at 3 dollars each and four stonemasons at 6 dollars did total the needed 33 dollars. However, when they used these amounts in the second scenario, they found that it did not work: four artists at 3 dollars each and three stonemasons at 6 dollars did not total the needed 37 dollars. They then assumed that the total for one artist and one stonemason was 10 dollars. Using the same logic, starting at 1 dollar per artist and 9 dollars per stonemason, then 2 dollars per artist and 8 dollars per stonemason, etc., they arrived at a solution of 3 dollars per artist and 7 dollars per stonemason, which they demonstrated would satisfy both requirements. In all three guess-and-check calculations, they assumed that the artists would earn less than the stonemasons would; so, they arrived at the correct figures for the solutions but had the assignments to the two types of workers backwards. They showed that four artists at 3 dollars each and three stonemasons at 7 dollars each would total 33 dollars. However, the original question stated that the cost of 33 dollars applied to three artists and four stonemasons. So, although their logic was correct, they made a minor error in the interpretation.

The second group, Figure 2, presented a tabular representation of the two scenarios. There are also several indications that they chose the values of seven and three dollars for the costs of the two types of workers, but there is no clear explanation of how they arrived at that conclusion. At the top left of the poster, four rows of seven marks each are made to represent the artists; each group of seven is circled, showing that the cost of each of four artists is 7 dollars. To the right, there are three rows of three marks each, representing that each of three stonemasons earns three dollars each. There is no indication that any values other than the correct solution were considered. There is also no indication of exactly how the correct values were calculated. However, at the bottom of the poster, two expressions are written: $\begin{array}{r} 4\times \\ +\,3\times\_ \\ 37 \end{array}$ and $\begin{array}{r} 3\times \\ +\,4\times\_ \\ 33 \end{array}$. These appear to indicate that an algebraic solution using two simultaneous equations was employed to arrive at the solution.

The third group, Figure 3, presented an addition solution using symbols to balance two equations. The top of the poster shows the two scenarios and the bottom of the poster shows the addition of these two. The top left of their poster shows three ten's (squares) and seven units (circles) to represent thirty-seven dollars. On the right of the equal sign, there are four A's, for artists, and three S's, for stonemasons. Directly below that is a similar configuration to represent a thirty-three dollar cost for three artists and four stonemasons. The lower left portion of the poster shows six ten's (squares) and ten units (circles) to represent seventy dollars. This is the addition of the ten's (squares) and units (circles) from the two equations on the top of the poster. On the right side, there are seven A's and seven S's, which are the sum of the A's and S's from the two equations. This group reasoned that they now had a total of seven artists and seven stonemasons and a total of seventy dollars. They circled one artist, one stonemason, and the group of ten units (circles) to show that one of each worker would cost ten dollars. This was one of the few groups who answered the question as written. They made no attempt to determine the individual costs for one artist or one stonemason. Members of this group were not unanimous about whether or not they should do so; however, several members of this group were confident that the question merely asked for the cost for one artist and one stonemason together and that individual costs were not required.

The fourth group lists the two scenarios and then depicts the artists making 7 dollars each and the stonemasons making 3 dollars each. Below that, the group lists their check work. This seems to be backwards. They reason that 33 dollars and 37 dollars added together equals 70 dollars; simultaneously, they reason that three artists and four stonemasons added to four artists and three stonemasons results in seven of each type of worker. If seven artists and seven stonemasons cost 70 dollars, the group reasons that one artist and one stonemason cost 10 dollars. An interesting approach to finding the individual cost for each type of workers follows. First, the group realizes that both scenarios have three artists and three stonemasons. Once scenario has an extra artist and the other scenario has an extra stonemason. Based on their conclusion that one artist and one stonemason cost 10 dollars, they derive that three artists and three stonemasons cost 30 dollars. Using this baseline, they argue that the scenario which has the extra artist is 37 dollars, which is 7 dollars more than their baseline. Therefore, the artist must cost 7 dollars. And, the scenario which has an extra stonemason costs 33 dollars, which is 3 dollars more than their baseline. Therefore, the stonemason must cost 3 dollars.

The fifth group, Figure 5, drew 37 hash marks on the left side of the paper and 33 hash marks on the right side of the paper. On the left side, there are three boxes drawn, each around three hash marks, and marked with an "s," for stonemason. And, the are remaining hash marks are separated into four groups of seven by being encircled; each is marked with an "a," for artist. On the right side, there are four boxes drawn, each around three of the hash marks, and marked with an "s," for stonemason. And, there are remaining hash marks are separated into three groups of seven by being encircled; each is marked with an "a," for artist. The poster, of course, only represents their final product and does not give insight to their thinking processes.

A review of the artifacts from the other three groups reveals a similar variety of approaches. Some groups used a strictly algebraic strategy with the simultaneous equations $4a + 3s = 37$ and $3a + 4s = 33.$ One group started off with finding ways to arrive at 37, including 25 + 12, 26 + 11, and 27 + 10, even though none of these contain one value which is divisible by 7 and another value which is divisible by 3. Their last attempt, though, 28 + 9, does factor correctly. They then used the same strategy to arrive at 33, using 19 + 14, 20 + 13, and finally arriving at the correct 21 + 12. Another group first wanted to determine who made more. They reasoned that the cost with an extra artist was greater than the cost with an extra stonemason, concluding that artists cost more. Using algebra, they found that the cost for one artist was 4 dollars greater than the cost of a stonemason. They then drew four boxes to represent four artists and wrote a "5" inside each one. They also drew three circles to represent three stonemasons and drew one hash mark in each one, because stonemasons make 4 dollars less than artists do. They computed the total, 23 dollars and saw that it was short of the required 37 dollars. They added a hash mark to each square and circle and added the four 6's and three 2's to arrive at a total of 30, again too low. Adding one hash mark to each square and circle again, they added the four 7's and three 3's to get the required total of 37. From their picture, it was clear to see that an artist earns 7 dollars and a stonemason earns 3 dollars.

An interesting observation made by one of the groups was that because the total cost in either scenario was odd and the number of total workers in each scenario was odd, then the individual pay for each type of worker must be odd. If there are four workers of the same type, then their total pay will be an even integer. But, the three remaining workers must have an odd wage or else the total cost would be an even integer. Using the same logic in the second scenario shows that both types of workers must have an odd value for their daily pay.

All five of the possible representation strategies were used by the groups: tables, pictures, graphs, numbers and symbols, and verbal descriptions.

The teacher reflections enabled me to focus on the understanding, reactions, and feelings of the individual teachers. While the posters showed how people in a group approached problem solutions in a variety of ways, the reflections gave me insight into how the individual teachers were feeling about the sessions, about their own competence, and about their classroom practices.

Several themes are present in the majority of the reflections. These are: the value of the struggle, the joy of using conceptual thinking, the importance of clarity, the advantage of building, the benefit of collaboration, and recognizing that there are multiple valid ways in which to approach problem solving, which leads to viewing student work with new eyes.

Teachers appreciated the value of the struggle for several reasons. Being forced to “figure it out” without reliance on rote procedures or “tricks” gave teachers a chance to think about their own thinking. Valuable discussions with their peers ensued. Understanding of concepts was developed. Several teachers reported “Aha” moments concerning ideas about rational numbers which they had formerly accepted but now actually understood, giving them a feeling of liberation. Teachers experienced frustration which made them more sensitive to the same feeling in their students; and, teachers saw the value in developing a thoughtful and defendable approach to problem solution. This is a skill which they want to transfer to their students. A teacher wrote, "I wish more classroom teachers fostered an environment where students can struggle with problems and work together to solve problems. Struggling through and listening to strategies of others has really opened up my thinking."

As the teachers' conceptual knowledge deepened, the teachers began to question their own knowledge and assumptions. Teachers gained such insight and expanded understanding through discussions that they want to incorporate more "talking about it" in their classrooms instead of heading straight for procedural solutions. Classroom discussions of problems and sharing solution strategies is seen as a valuable approach both to clarify problems for our students as well as to develop their conceptual thinking.

The cathedral problem provided a poignant example of a question which is easy to misinterpret. The question asked, "What would be the expense of just one of each worker?" Whether the answer to the question is "$10" or "$7 for an artist and $3 for a stonemason" may never be resolved. Even after the classes discussed the idea that the question did not ask for the individual rates, some teachers were convinced that the question required the rates for both workers. The important point is that the question was, apparently, open to interpretation. Teachers, both in reflections and in verbal commentary, noted that they learned to be very clear when they write questions. One teacher commented to me that she was going to review all of her assessments to ensure that she did not have any "open to interpretation" questions. She showed some angst in saying that she hoped she had not done that to her students in the past.

Teachers reported that the reasoning up and down helped them to break problems into chunks and build on those chunks. They saw how building on known concepts or known quantities gave them a sense of control as opposed to the lost feeling we sometimes experience during the introduction of a completely new idea. The teachers realize that the latter is a source of concern, frustration, and fear in their students. One teacher commented that she never realized how emotional the process could be and that she was gaining a new perspective on her students and how she interacts with them. Another teacher wrote that she would use reasoning up and down to help her students focus on what they already know and then guide them in building on that knowledge. Several teachers remarked on the importance of labeling processes so that students have a clear picture of how the concepts tie together; this leads to the development of conceptual understanding and the internalization of concepts and processes for the students.

Teachers appreciated the collaborative nature of the institute. No one felt as though they were left to fend for themselves with no help. Struggling through problem solutions with colleagues, analyzing their approaches, questioning their reasoning, and, contributing to group efforts were noted by the teachers as being very beneficial to their discoveries during the week. Meetings with colleagues have already been planned by several of the teachers to discuss their progress and to choose problems to incorporate the reasoning up and down strategies into their curricula before the start of the 2010-2011 school year.

So often an approach to a mathematical problem is formulaic, totally plug and play, and without much attention given to concepts. After all, adding ½ and ½ and getting ¼ does not make any sense if one would only take a few seconds to think about it. The emphasis of thinking, really thinking, about a problem before rushing to get a solution was a major issue for the teachers. For example, a problem stated that 1 robot can make 1 car in 10 hours and asks how long it will take 10 robots to produce 10 cars. It is far too easy to slip into the mindset that every value in the first scenario has been multiplied by 10. Almost every group had someone do this. However, when that teacher looked at the answer of 10 hours, the realization that the answer made no sense came quickly. One teacher looked at her work, gave a quizzical look, and said, "now, that just can't be right. Hmm, why doesn't that work?" Then, the group thought about the problem and the relationship between the values. The teachers recognized the crucial importance of thinking about the question before crunching numbers. Additionally, as can be seen in the posters, the teachers gained an appreciation for the validity of multiple approaches to problem solution. Several teachers mirrored that idea in their writings.

Coupling these two ideas of really thinking and multiple valid approaches, teachers recognized that they need to take a closer look at their students' work. One of the class discussion examples, studied in the unitizing lesson, showed a teacher's comment that a student was adding denominators. The teacher had failed to see how the student had validly defined the unit; the student's work was correct. No one in the discussion groups ridiculed the teacher's mistake. There was a purposeful silence, then facial expressions of dread and low, hushed mumblings of "oh, no, I hope I've never done that." One teacher broke the silence by interjecting that this lesson would help all the teachers recognize that there are many different correct methods and that we should all find what the students have done correctly, even if their answers are wrong. Another teacher wrote that it was important for her students to feel comfortable demonstrating their knowledge in different ways and that this helps the teacher to know what the student does understand. Lastly, another teacher reflected, "I am also starting to think differently about analyzing student work. When problems have the opportunity of yielding a variety of correct answers, it is important to consider what the student is doing and what math they can do and understand."

Conclusions

My class observations, conversations with participants, review of the team posters and the individual reflections have highlighted several central ideas.

Teachers need the opportunity to struggle with problems in order to develop deep understanding of rational numbers. While many teachers expressed frustration with the homework problems as well as the in-class problems, they also recognized that their frustration led them to think about rational numbers in ways which they had not employed previously. This led to deeper understanding. Several teachers reported that they now "get" rational numbers and are gaining appreciation for the connections between concepts; they attribute this to the experiences of struggling through the investigative problems without the crutch of plug-and-play procedures.

The daily investigations, such as the cathedral problem, led to discussion and exploration of much more than simply trying to find an answer. Teachers questioned each other's thinking and would not allow unsubstantiated assumptions. The focus was on mathematical reasoning, not the answer. I repeatedly heard teachers asking each other, "please explain that again, I don't understand where you are going with this" or "why would that be reasonable way to solve this?" Knowing that numerous approaches to problem solution were both possible and valid freed the teachers to concentrate on the soundness of their approaches, resulting in the teachers being able to develop more profound understanding.

Mistakes and confusion allowed the teachers to use mathematical reasoning and arguments to do side-by-side comparisons of solutions, or just talk through comparisons of solutions to find where they did not match up. Then, the teachers would strategize to determine not only how to proceed but also to determine why one method did not work. For example, "1 robot can make 1 car in 1 hour" does not mean "2 robots can make 2 cars in 2 hours." Teachers discussed why a simple "multiply through" technique did not work. Teachers benefitted from these discussions in several distinct ways. First, they began to see that real problems involving rational numbers are not simply plug-and-play exercises; they are multi-layered challenges which require analysis, sound reasoning, and understanding of the relationships among quantities. Second, they recognized the profound importance of conceptual understanding as a baseline for strategizing approaches to problem solving. And, third, they gained an acute appreciation for the frustration of their students who apply incorrect procedures and cannot understand why their answers are incorrect.

Teachers are learning to think more insightfully and to use reasoning in ways which they have not previously employed. This cannot be expected to occur overnight. The same is undoubtedly true for our students. It takes time and practice before the teachers may see connections between the concepts and techniques which they learned at the summer institute. While there was no appreciable difference in the approaches to the cathedral problem on Friday than there were earlier in the week, this may not be true when the participants return in October. By then, they will have not only had the opportunity to develop their mathematical thinking but also to use the techniques of the course in their classrooms.

As an observer, I found that I consciously had to restrain myself from falling into teacher mode. This allowed me to gain perspective on how other mathematics teachers think, how they approach solutions, and how they utilize their mathematical understanding. At times, I was somewhat surprised by the level of misunderstandings which I observed, but I was also surprised by the unique approaches which were sometimes utilized, approaches which I probably would not have devised or even considered.

Stress was sometimes palpable. Teachers worried about their grades for the course, how to complete a concept map, what to include in their course reflection, etc. Several anxious teachers approached me for assistance throughout the week.

If a particular technique is not to their liking, or they have difficulty with it, the teachers tended to dismiss it: “oh, not that again,” “I don’t like that,” or “I don’t understand that, let’s use something else.” Kids do the same thing, only kids may not know that there are multiple valid approaches to problem solving.

Teachers reaped great benefit from the course and most acknowledged that. Some focused on how much their own understanding had expanded; some focused on how their classroom approach would improve; and, some marveled at how their own eye-opening week would benefit both themselves and their students.

Implications for Teaching and Learning Mathematics

As concluded by the teachers, conceptual understanding is the bedrock to advancement through any mathematics curriculum. K-8 teachers who themselves do not have a deep understanding of foundational mathematical concepts, perhaps most importantly, rational numbers, will be unable to develop deep understanding in their students. Therefore, it is imperative that our teachers have deep understanding of the concepts which they teach.

The professional development at the summer institute was centered on the deepening of understanding of rational numbers, the development of analytical skills and mathematical reasoning, and, a commitment to inquiry. Part of the stated goal of the project is the development of mathematical teaching knowledge through a collaborative network and the promotion of algebraic conceptual thinking. So, is this the roadmap to increased understanding of rational numbers? How do mathematics teachers develop deep understanding of rational numbers? And, how can professional development for K-12 mathematics teachers be designed and implemented so that the teachers develop deep understanding of rational numbers?

The teachers themselves have voiced their beliefs that the struggles, the collaborations, and the iterative process have opened their eyes, allowing them to: discover how rational numbers behave; think about rational numbers and their properties in ways which they had not previously considered or experienced; and, to question their own beliefs about teaching rational numbers.

So, it follows that professional development for K-8 mathematics should include activities which employ these strategies, as the summer institute did. Albeit, after only one week, it is not realistic to declare that the teachers who participated have developed a deep understanding of rational numbers. That may be more fairly evaluated after the teachers have had time to reflect on their learning, incorporate their new knowledge into their classroom teaching, and, evaluate their own experiences and their students' experiences. However, the level of improvement and enlightenment which the teachers reported in their reflections is noteworthy.

The summer institute did not, of course, simply follow a script in autopilot. The discussions, which were such an integral part of the teachers' learning and discovery, were skillfully managed by the instructors. The instructors were not merely bystanders; they engaged the teachers in productive conversations, listened to what the teachers' were saying, and guided the discussions, as needed. The instructors encouraged the teachers to share their ideas and kept the discussions focused on mathematical thinking, all while ensuring that professional courtesy was maintained. The teachers, in turn, will need to ensure that their classroom discussions also stay focused on mathematical thinking and the expression of different ideas and approaches.

The institute was designed and implemented purposefully, matching the activities with the intended goals. No time was wasted or spent on trivia. All of the mathematics problems used were chosen with the specific goals of engaging the teachers, eliciting productive thought and discussions, and, deepening the teachers' understanding of rational numbers.

In only a week's time, the project was able to enlighten the teachers in several different ways. Teachers realized that they need to recognize that multiple, differing approaches to solving a particular problem may be valid. Exposure to the thinking processes of fellow K-8 mathematics teachers illustrates this concept dramatically. If a team of K-8 mathematics teachers devises multiple different approaches to a particular problem, it seems reasonable, even predictable, that the same will be true of our students. The focus needs to be on the thought process and tying concepts together. We want our students to develop correct and useful number sense. We want them to understand how quantities are interrelated.

Teachers get frustrated when they do not know what to do, an uncomfortable feeling for the "knowledge authority." Several teachers remarked to me that "now, I know how my students feel." This same remark was present in numerous reflections as well. Teachers who experience the frustration of struggling will be more acutely aware of it in their students. Hopefully, with collaboration and focus on conceptual understanding, frustrations will be minimized.

Collaboration is not only a great stress reliever but also a rich learning environment, as long as everyone in the group contributes. Several teachers remarked that they will use student groups in their classrooms more frequently because they saw the benefit of such strategies for their own work. I saw many instances of teachers asking others in their group to explain their reasoning. This helped the teachers to understand their colleagues' reasoning and also helped the speaker clarify her/his own thoughts through explanation. Then, the table discussed whether or not the logic was valid and if they wanted to use that approach.

Finally, patience and perseverance may be the keys. Looking at the posters of the cathedral problem, I thought I would see progression through the days, but I did not. On the fourth day, the "guess and check" method was still being used. The days of the institute were full and exhausting. The teachers completed work on different strategies each evening. Undoubtedly, they focused on the work of the day and did not have much time to think about making connections. This takes time. A few of the reflections included statements concerning how a teacher was beginning to see how the different concepts which were discussed could be tied together. But, the majority of the teachers had not arrived at the realization by the end of the week. What they did appear to all grasp was that the goal is thoughtful reasoning, not the correct answer, and that there are usually multiple, varying approaches with are mathematically sound. Even though faulty reasoning will sometimes lead to the correct answer, it is not satisfying, nor does it reap any real benefits for the student. Correct reasoning and execution will always lead to the correct answer. So, focusing on what the students are doing correctly, building on what the students already know, guiding our students to think and be analytical, asking if the steps and the answer make sense, and being patient as our students struggle through partially correct approaches, will build the students' confidence, develop critical thinking skills, and, lay a solid foundation for high school mathematics and thriving in the mathematics of everyday life as an adult.

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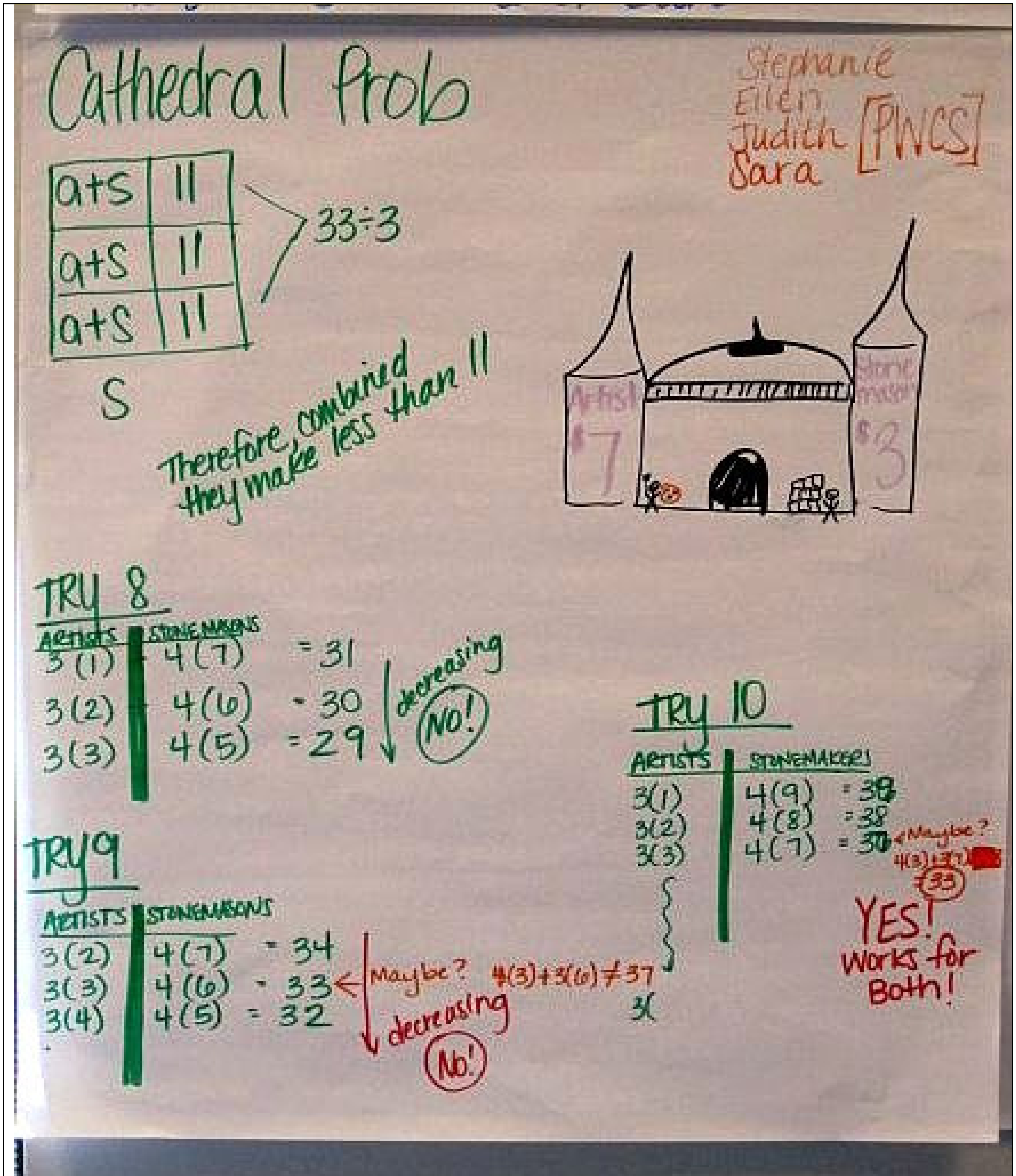
Appendix A

The Cathedral

While building a medieval cathedral, it cost 37 guilders to hire 4 artists and 3 stonemasons, or 33 guilders for 3 artists and 4 stonemasons. What would be the expense of just 1 of each worker?

From Burns, S. (Ed.). (2003). September's Menu of Problems. Mathematics Teaching in the Middle School, 9, 32-36.

Appendix B



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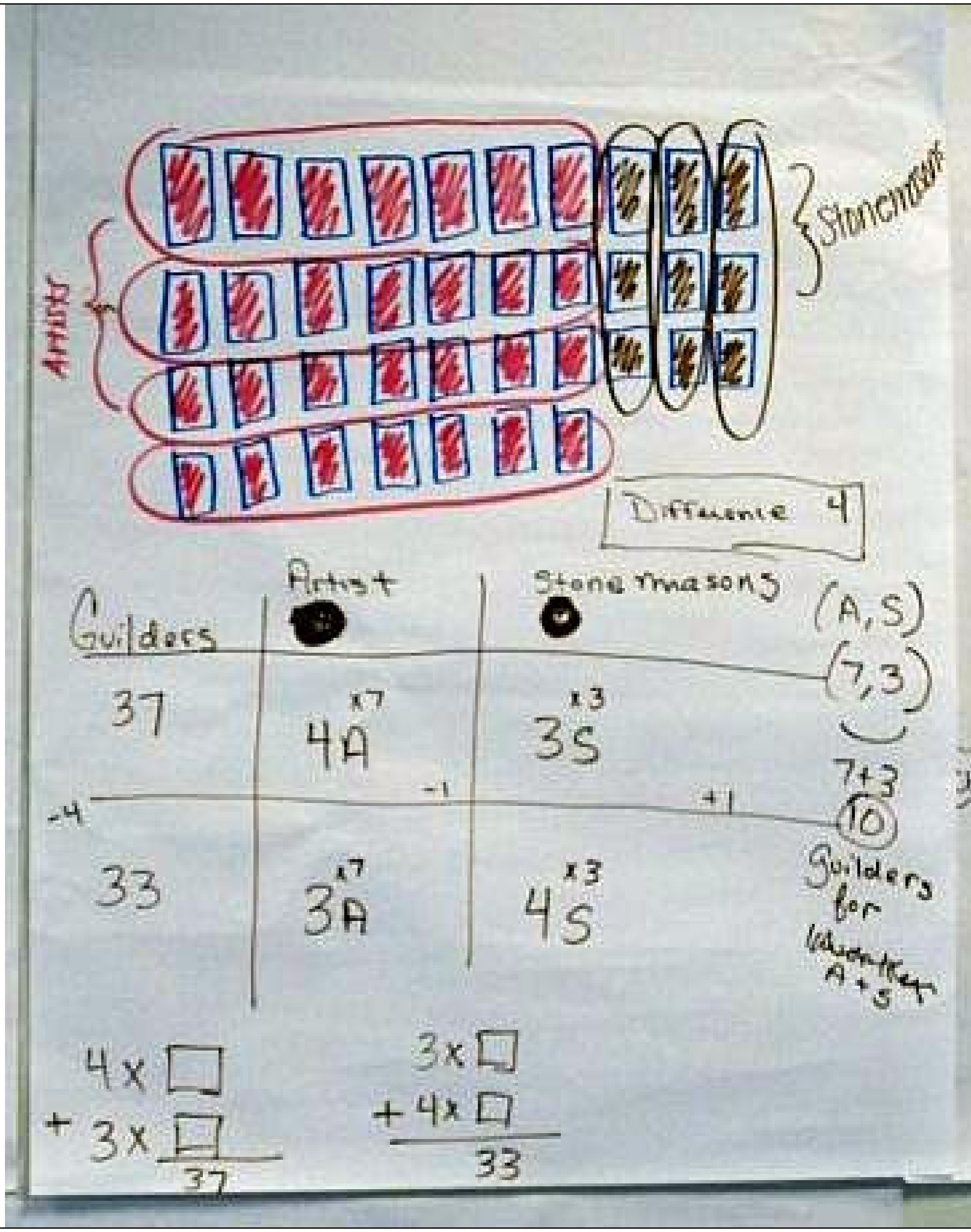

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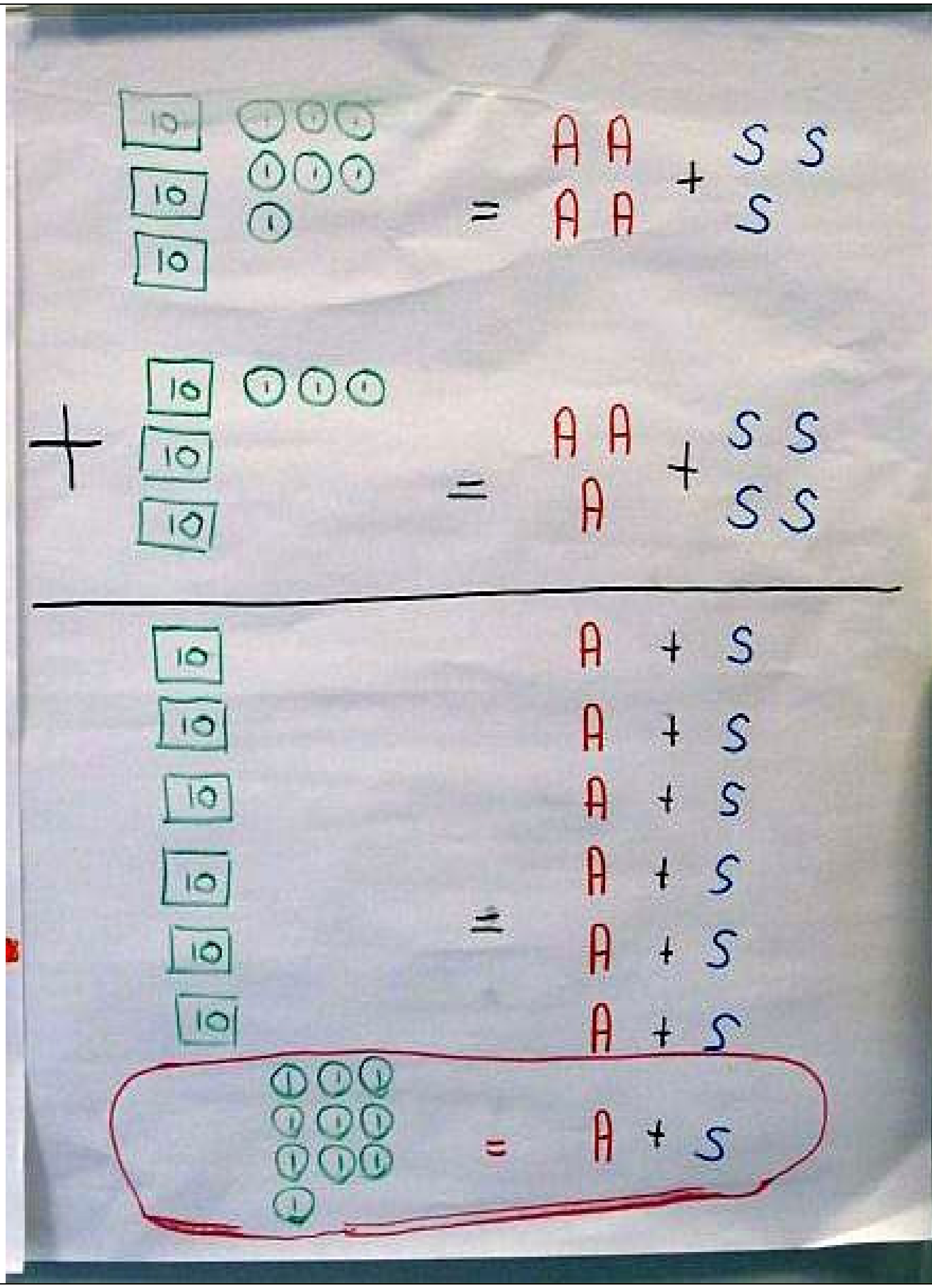

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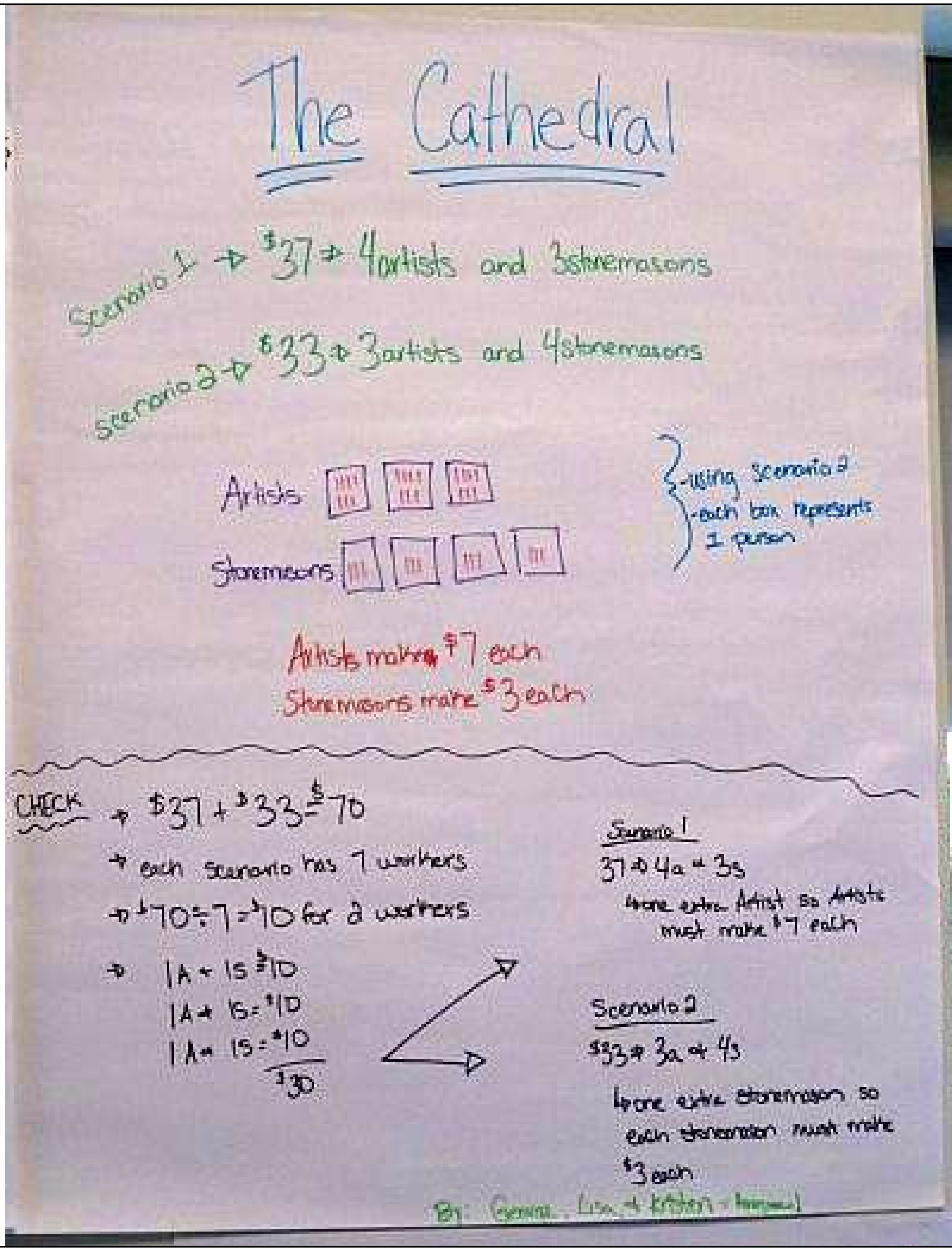

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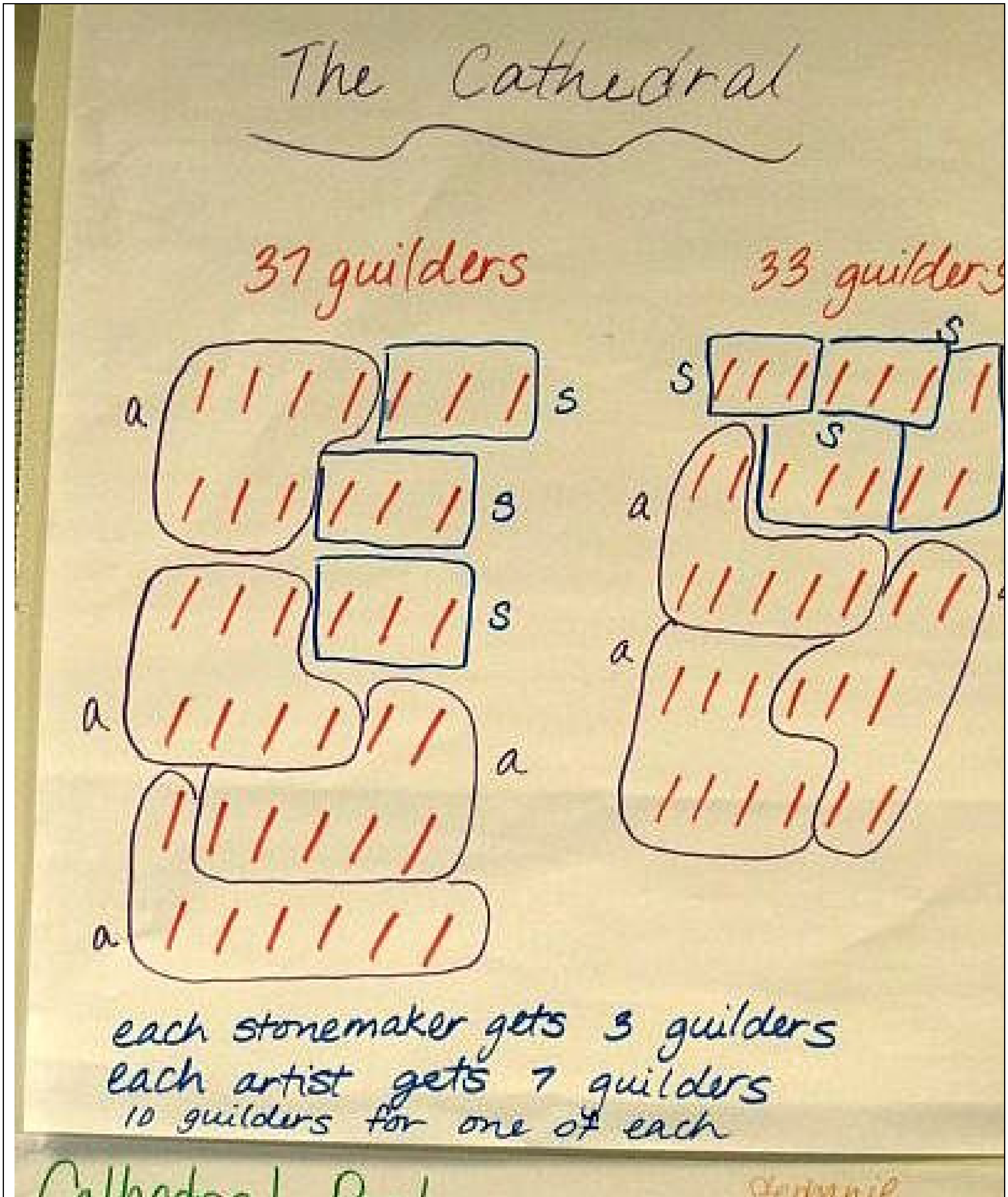

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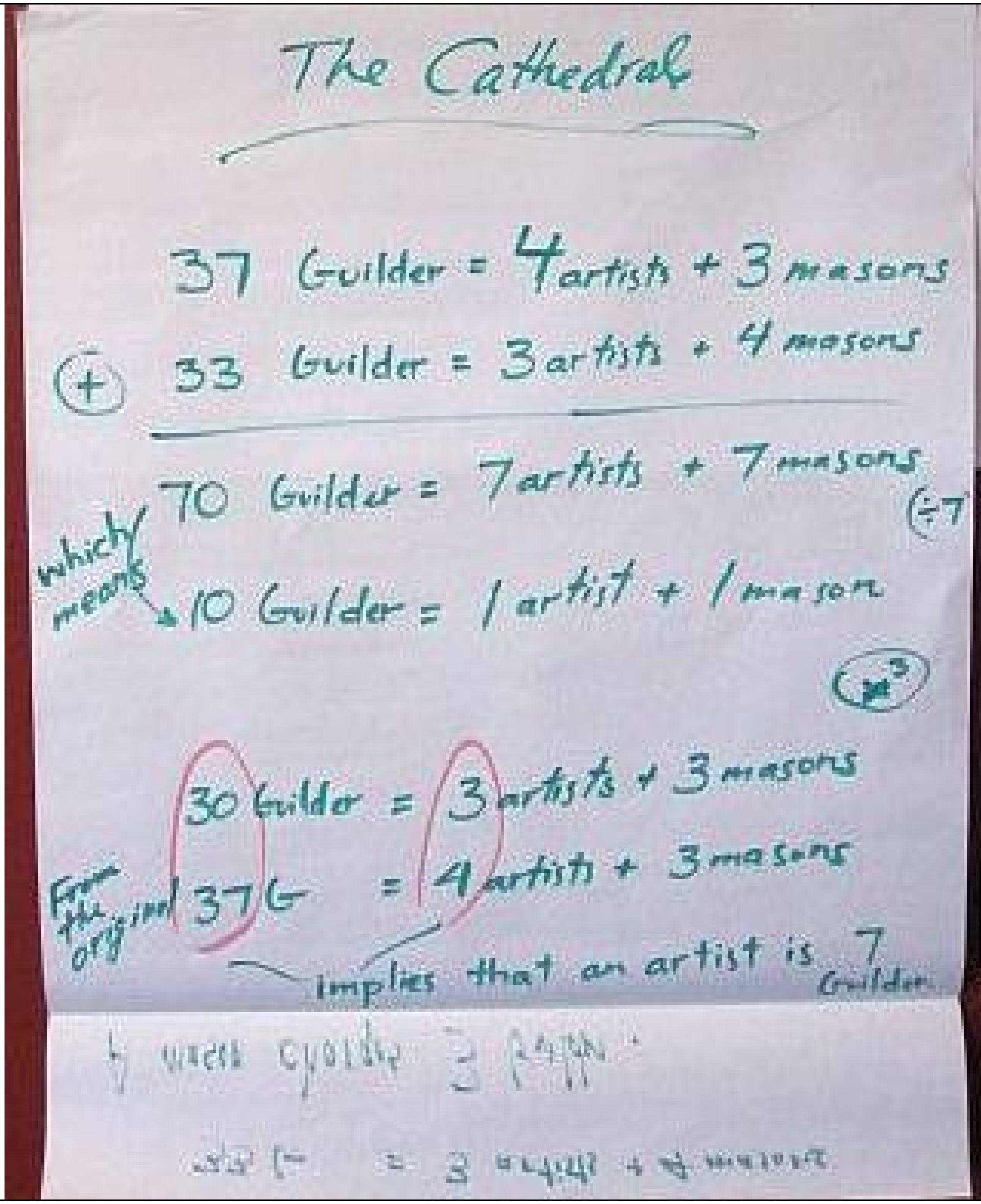

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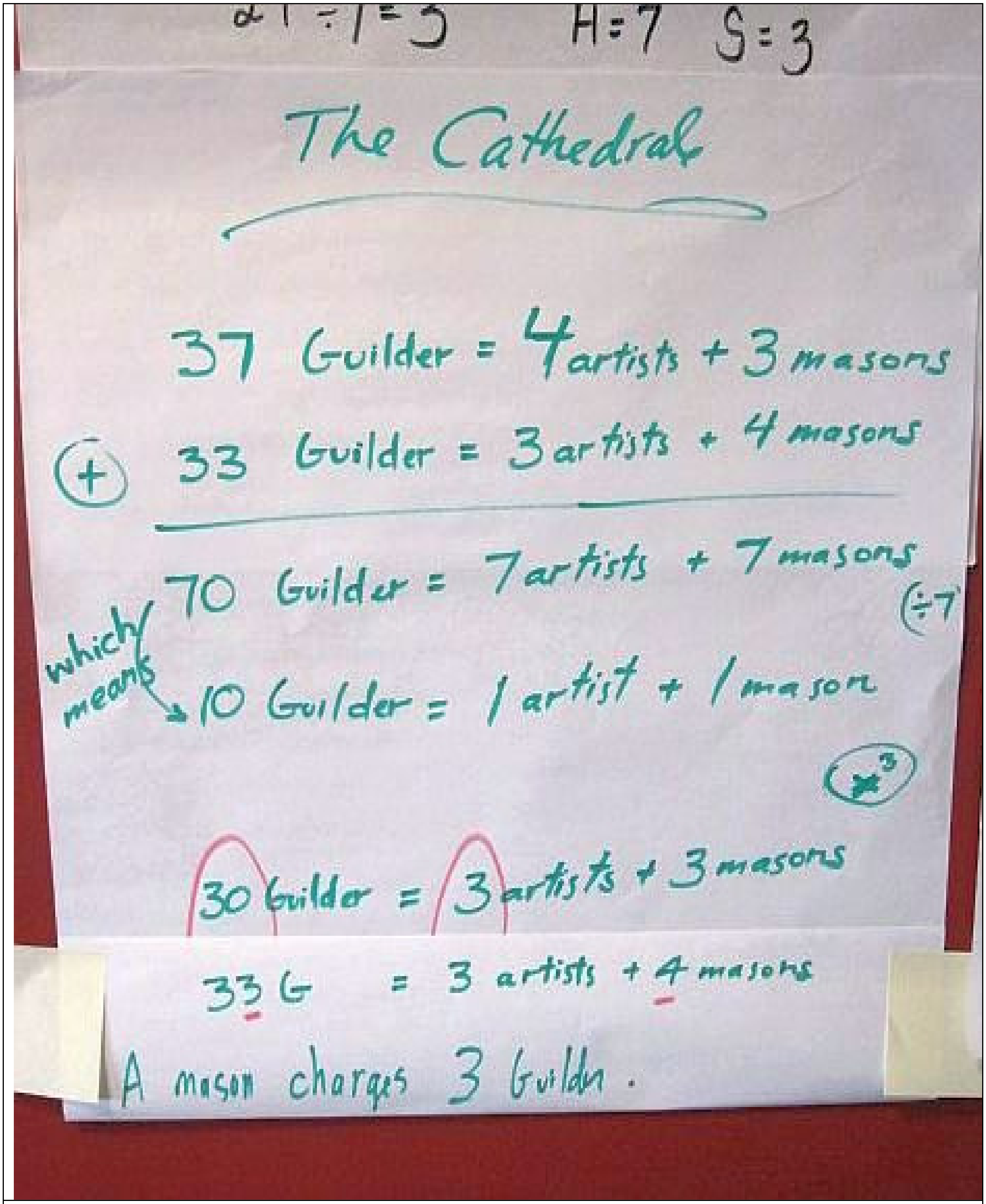

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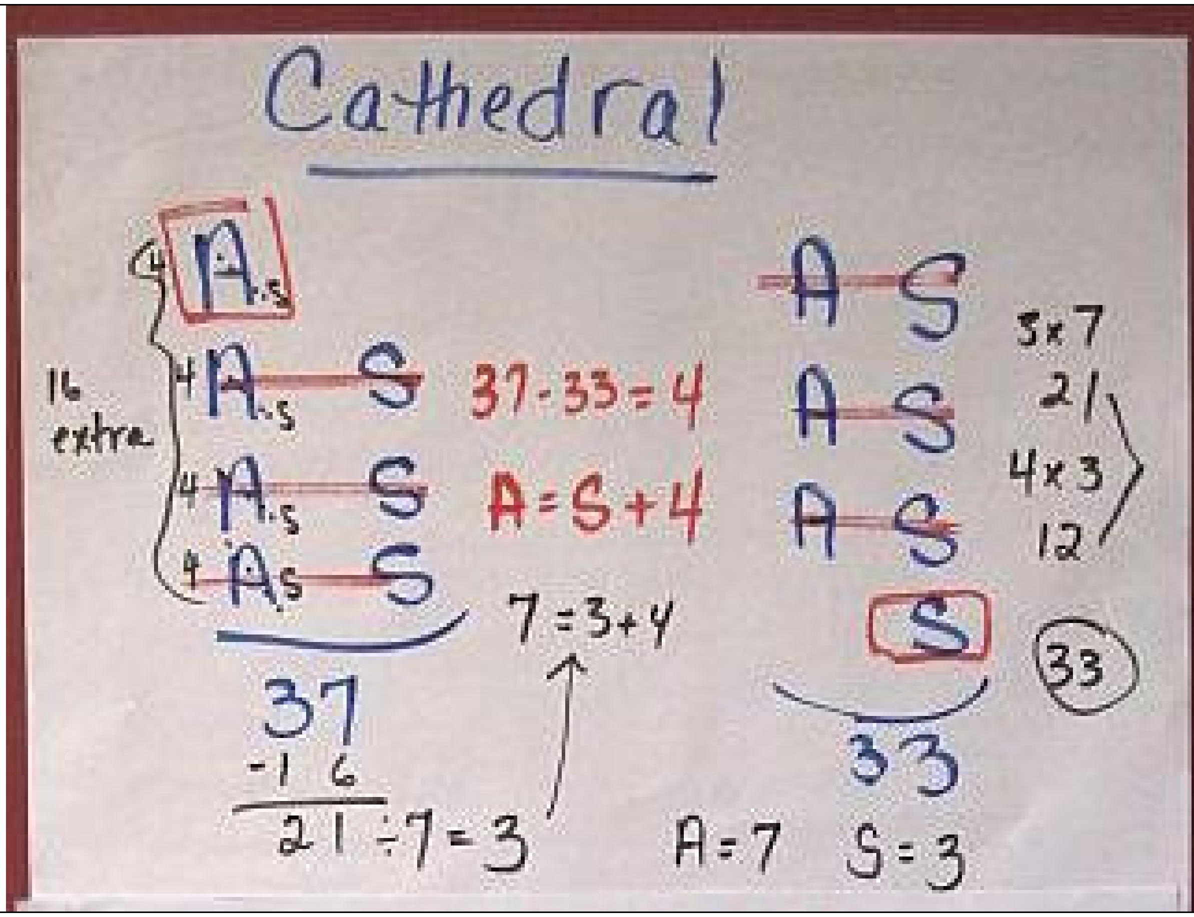

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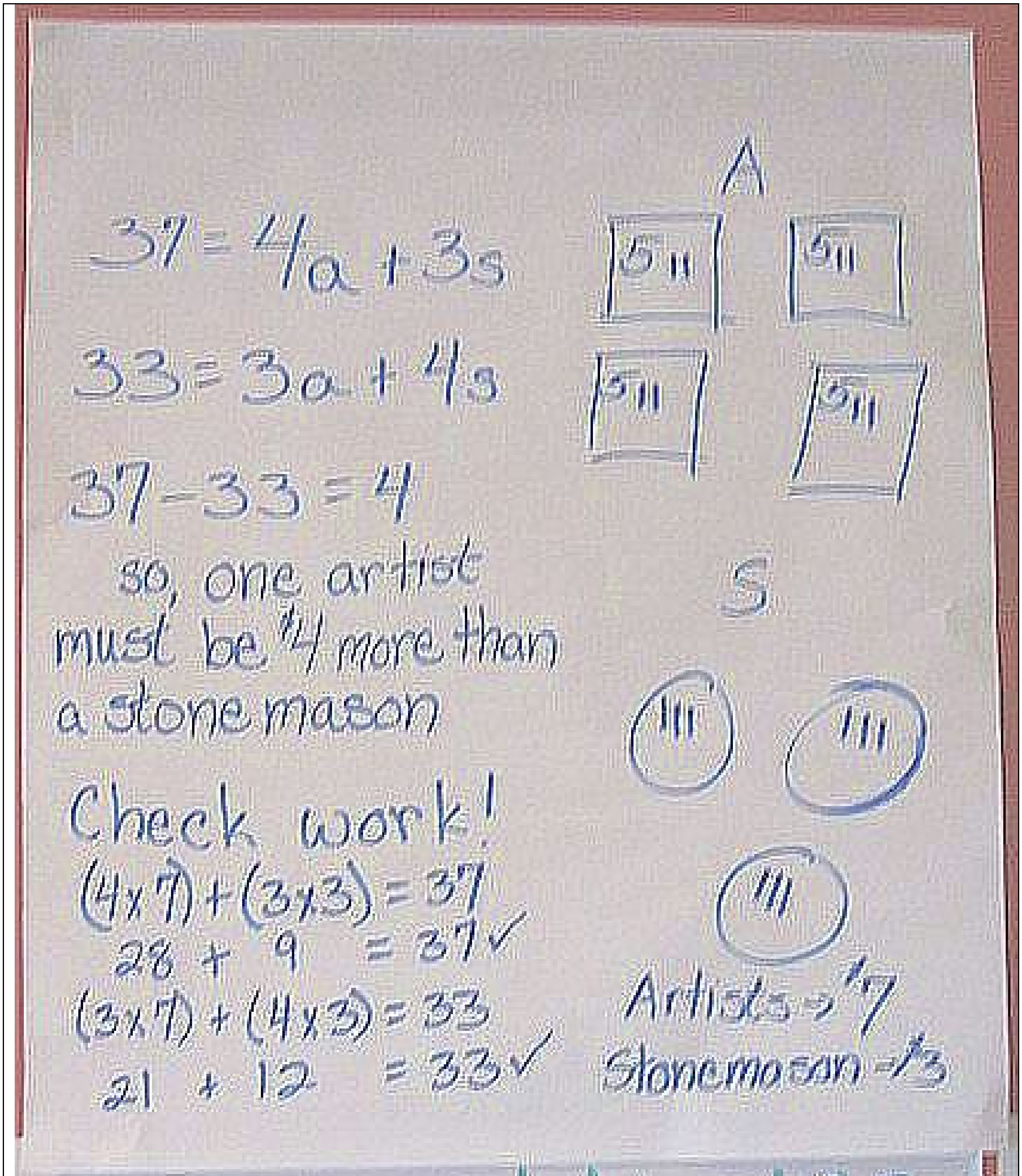

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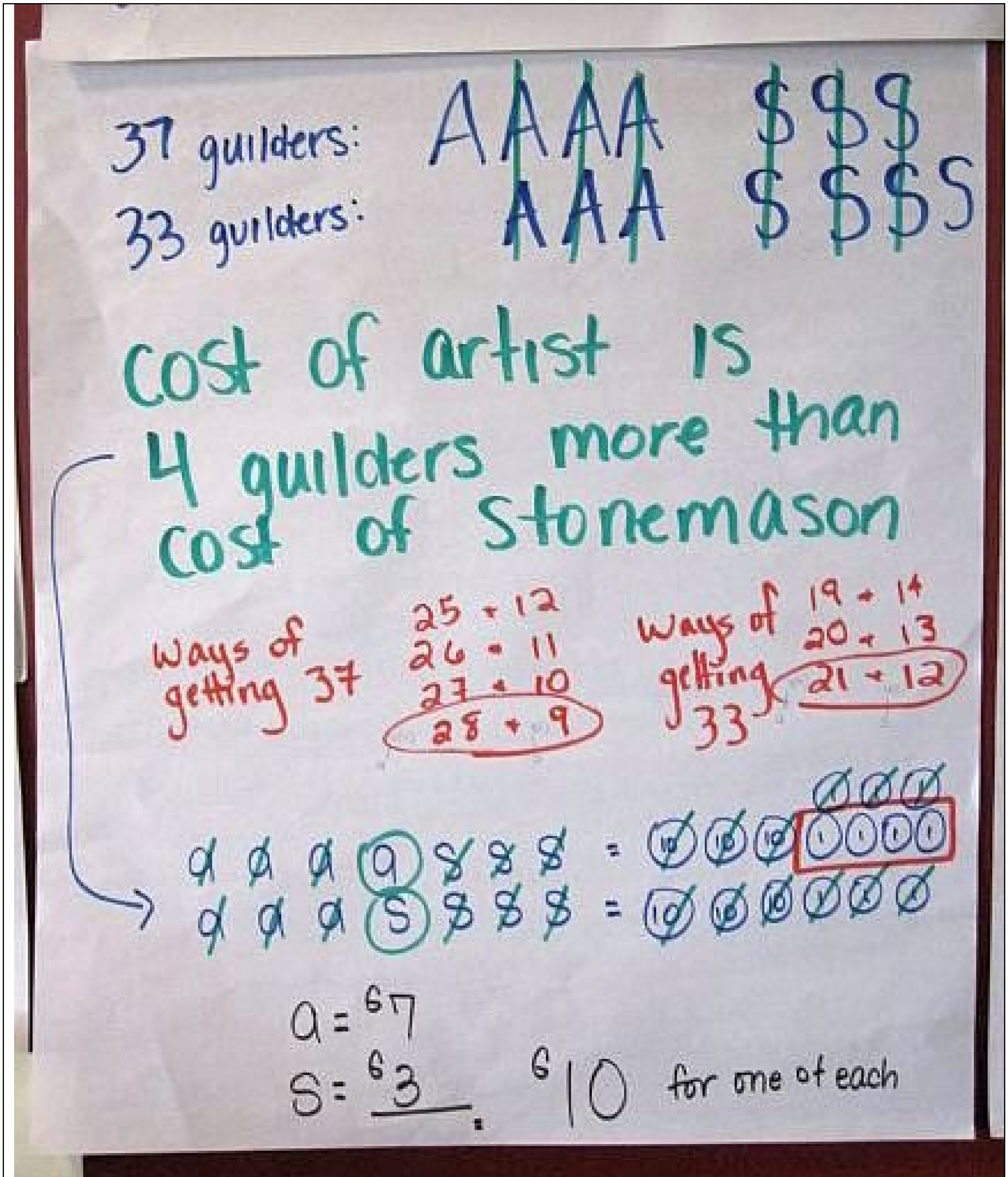

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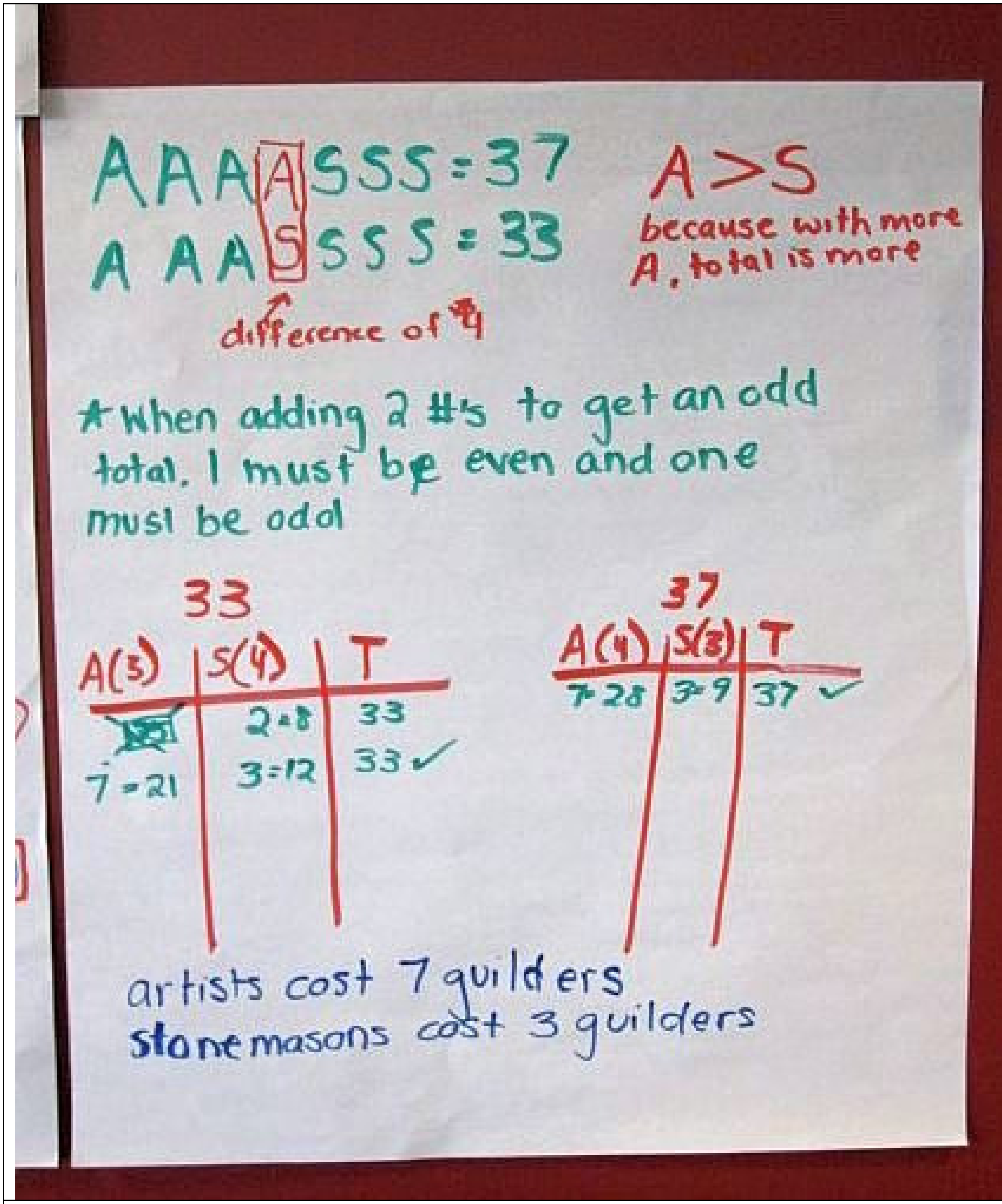

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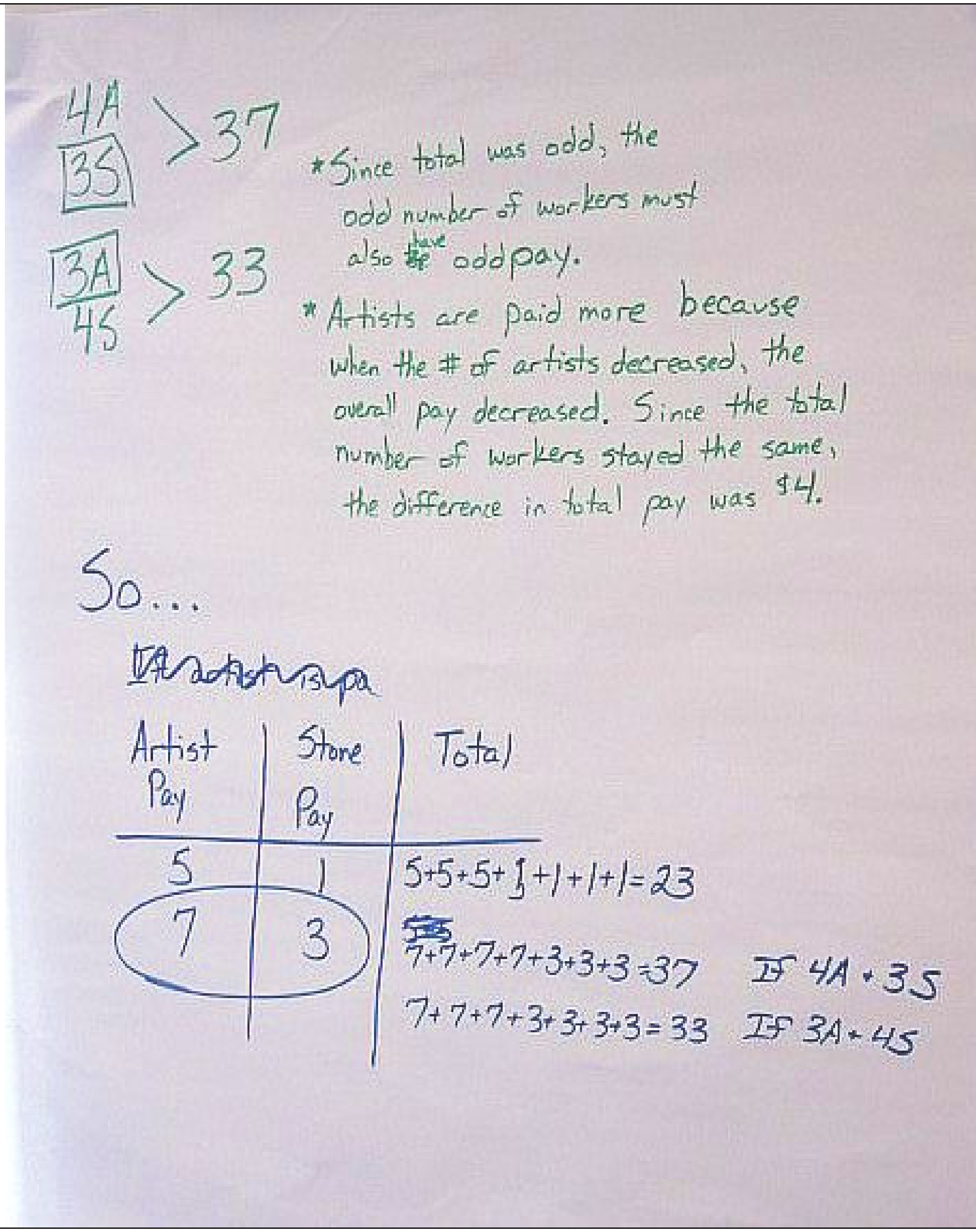

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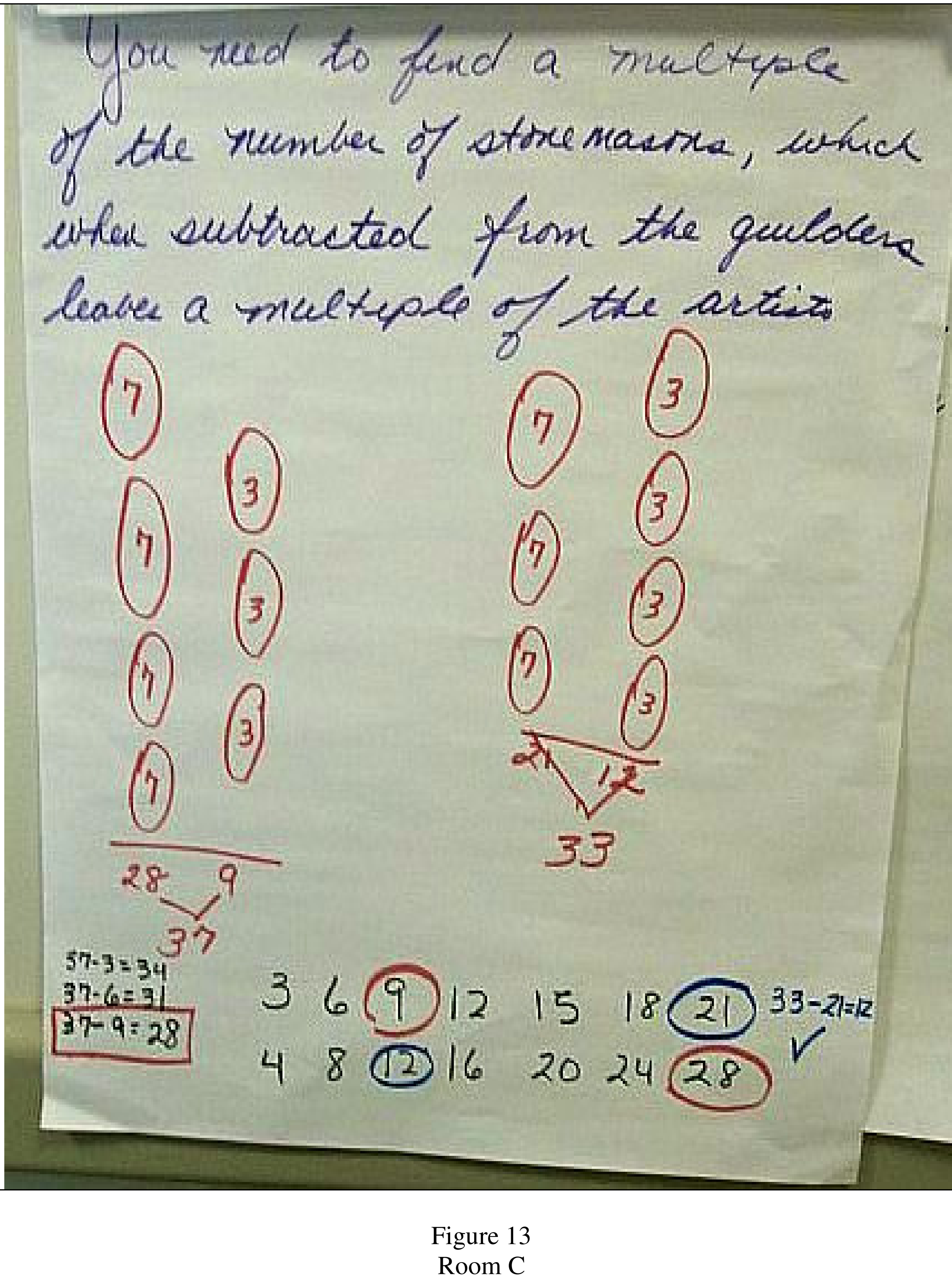

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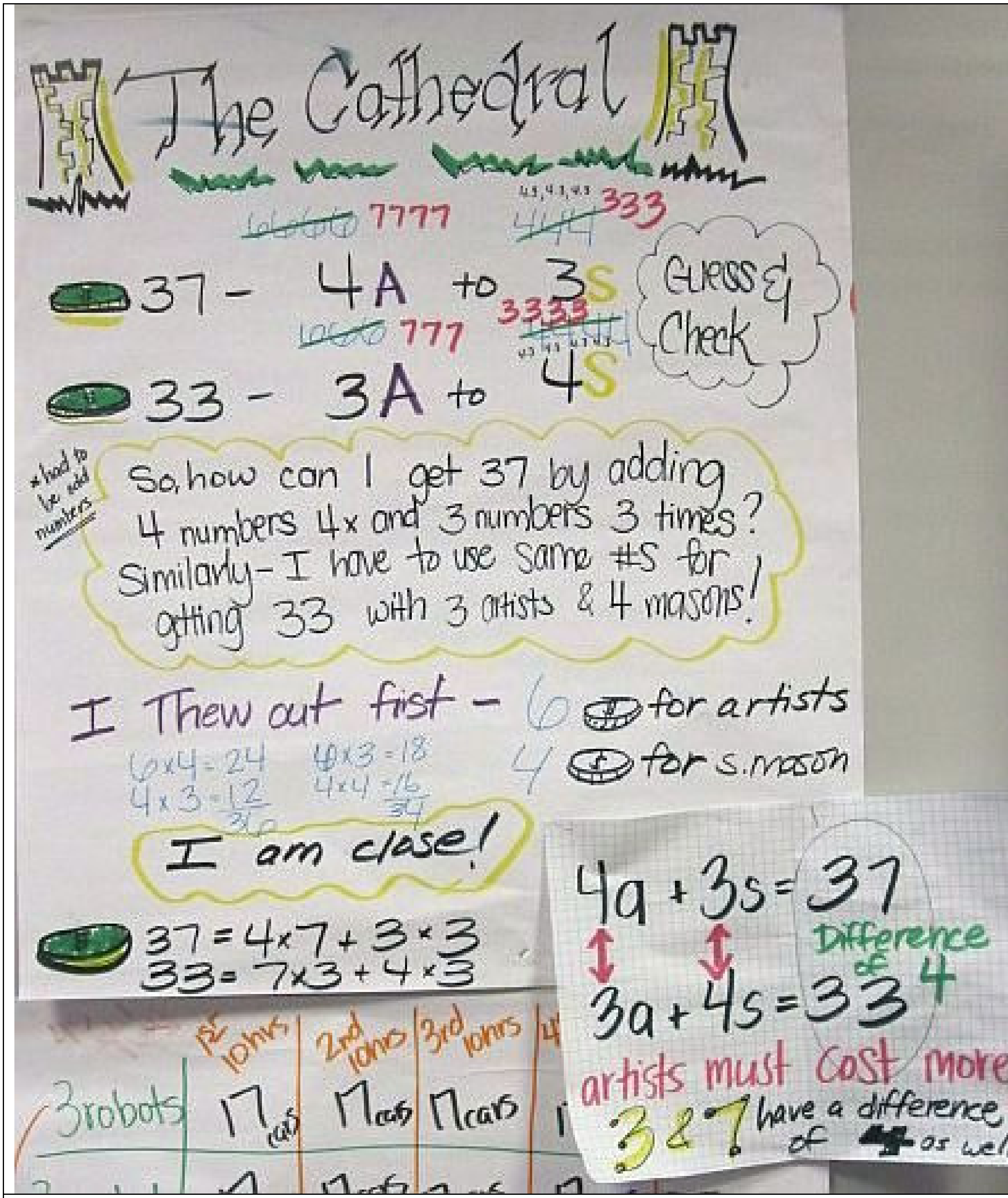

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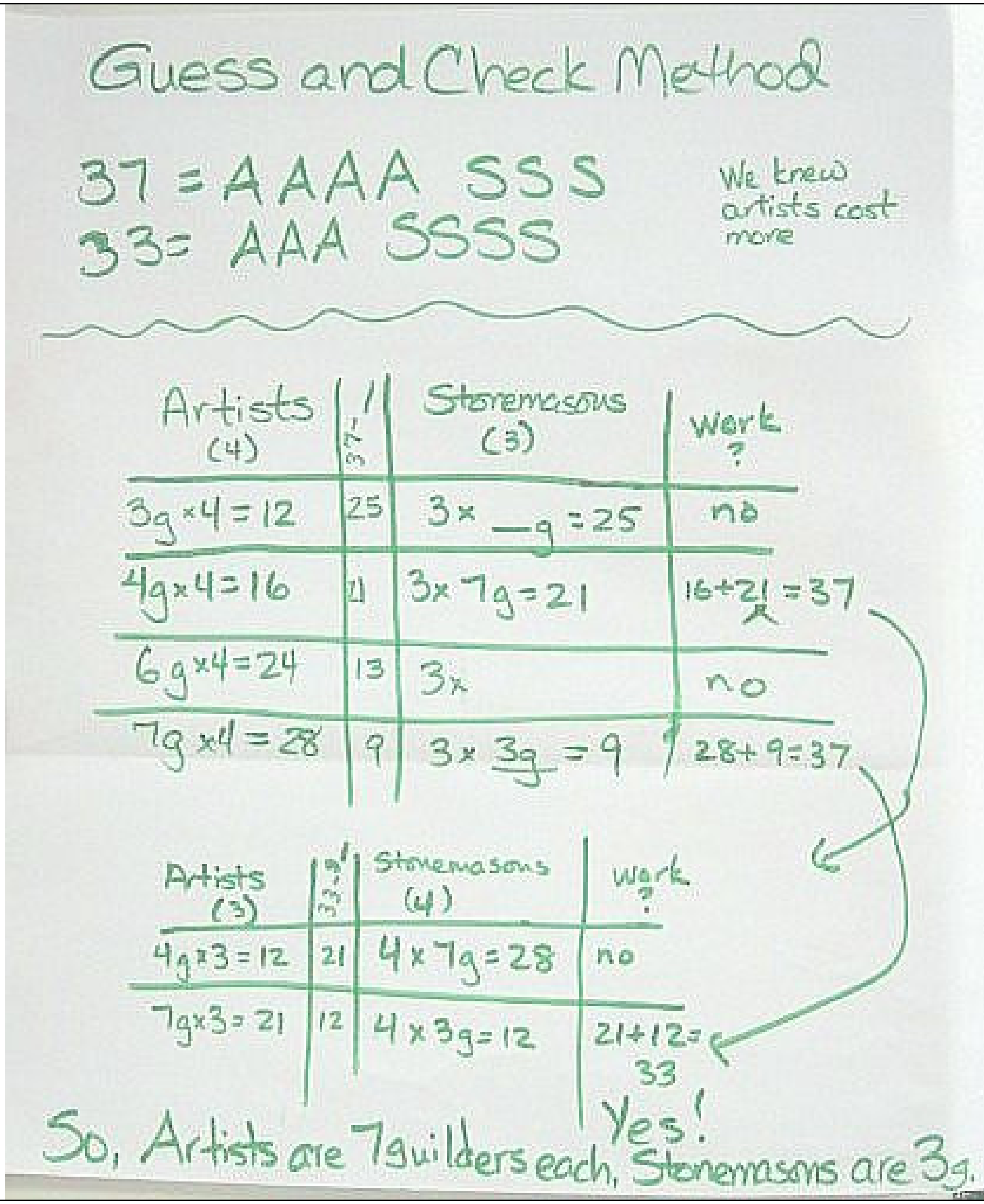

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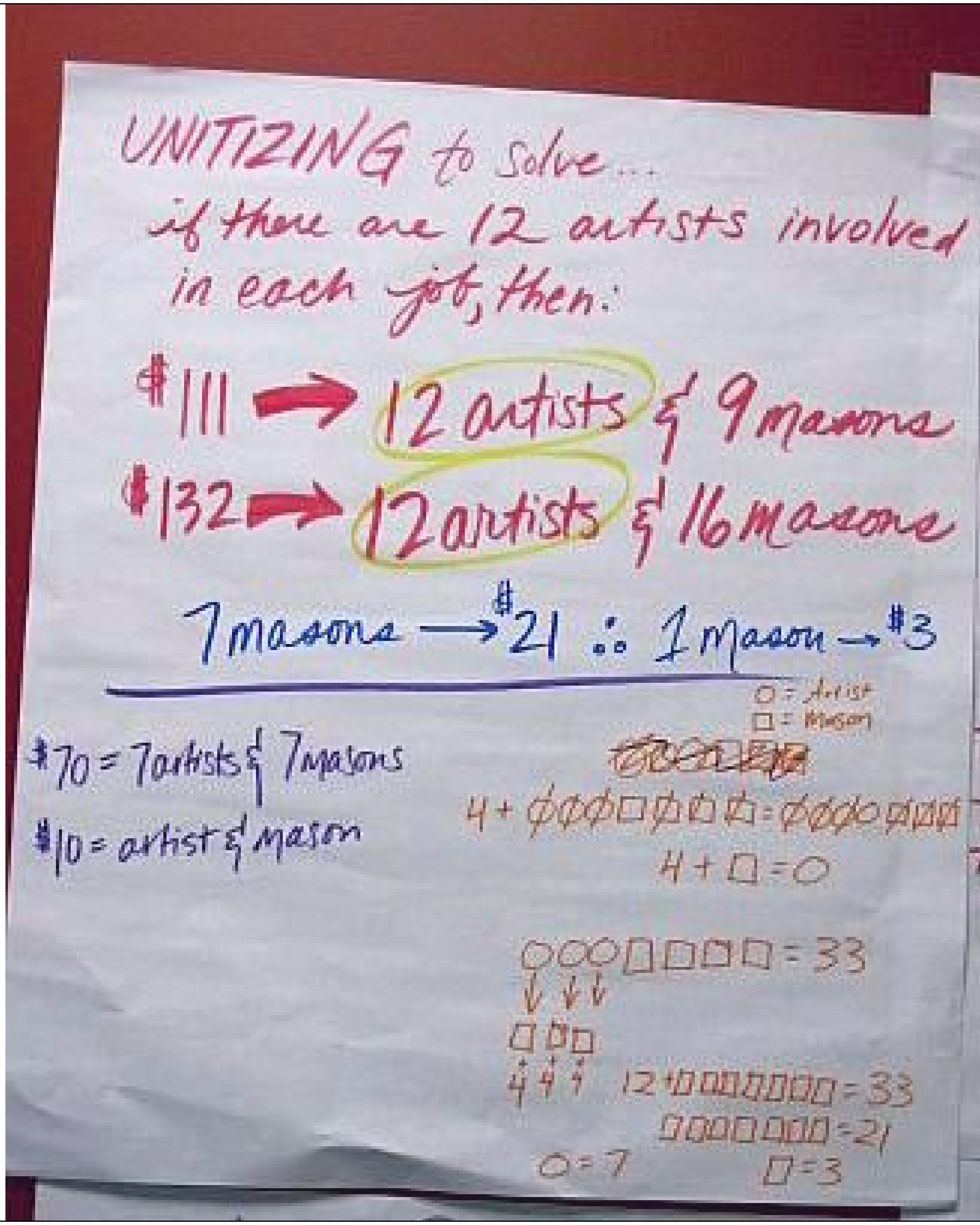

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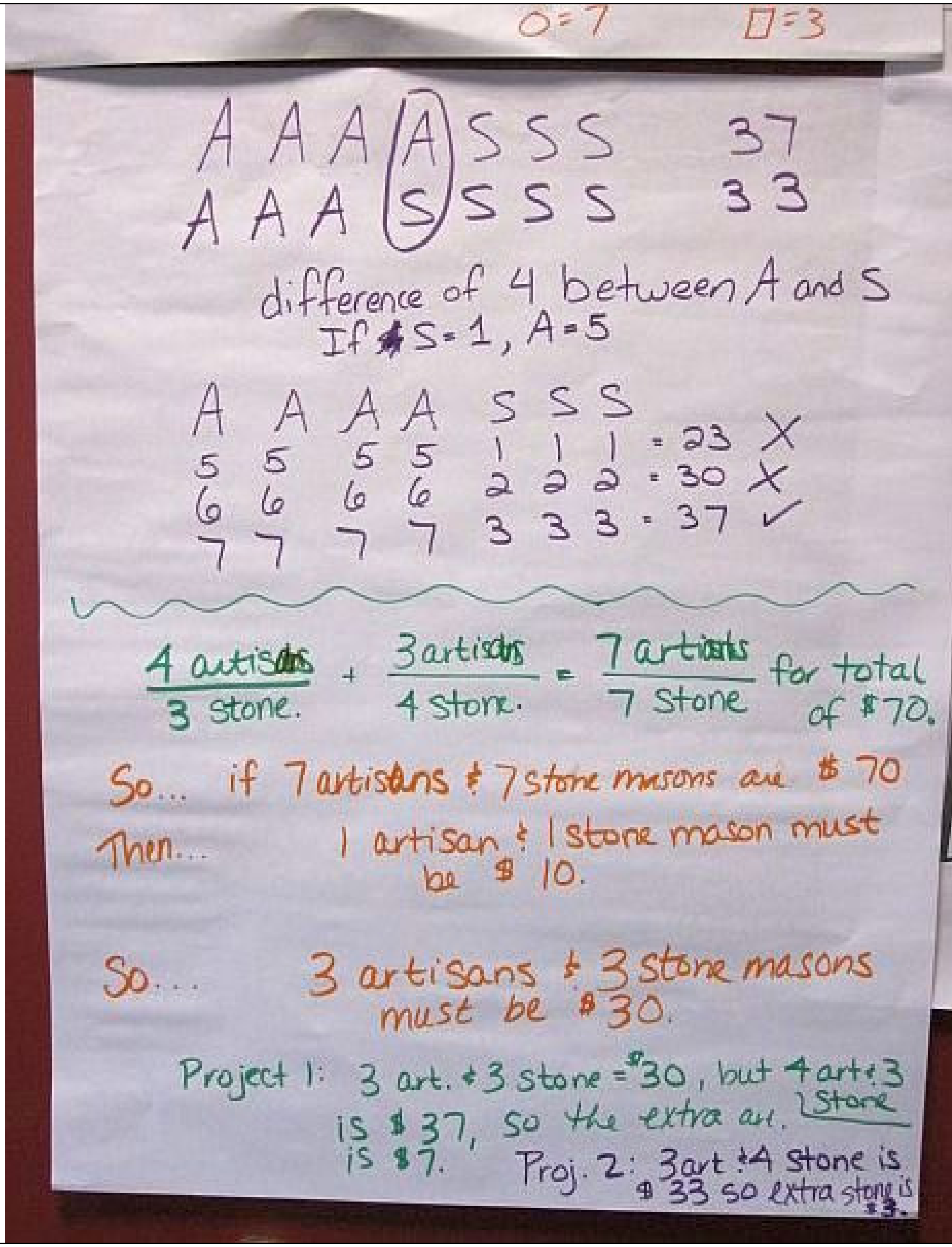

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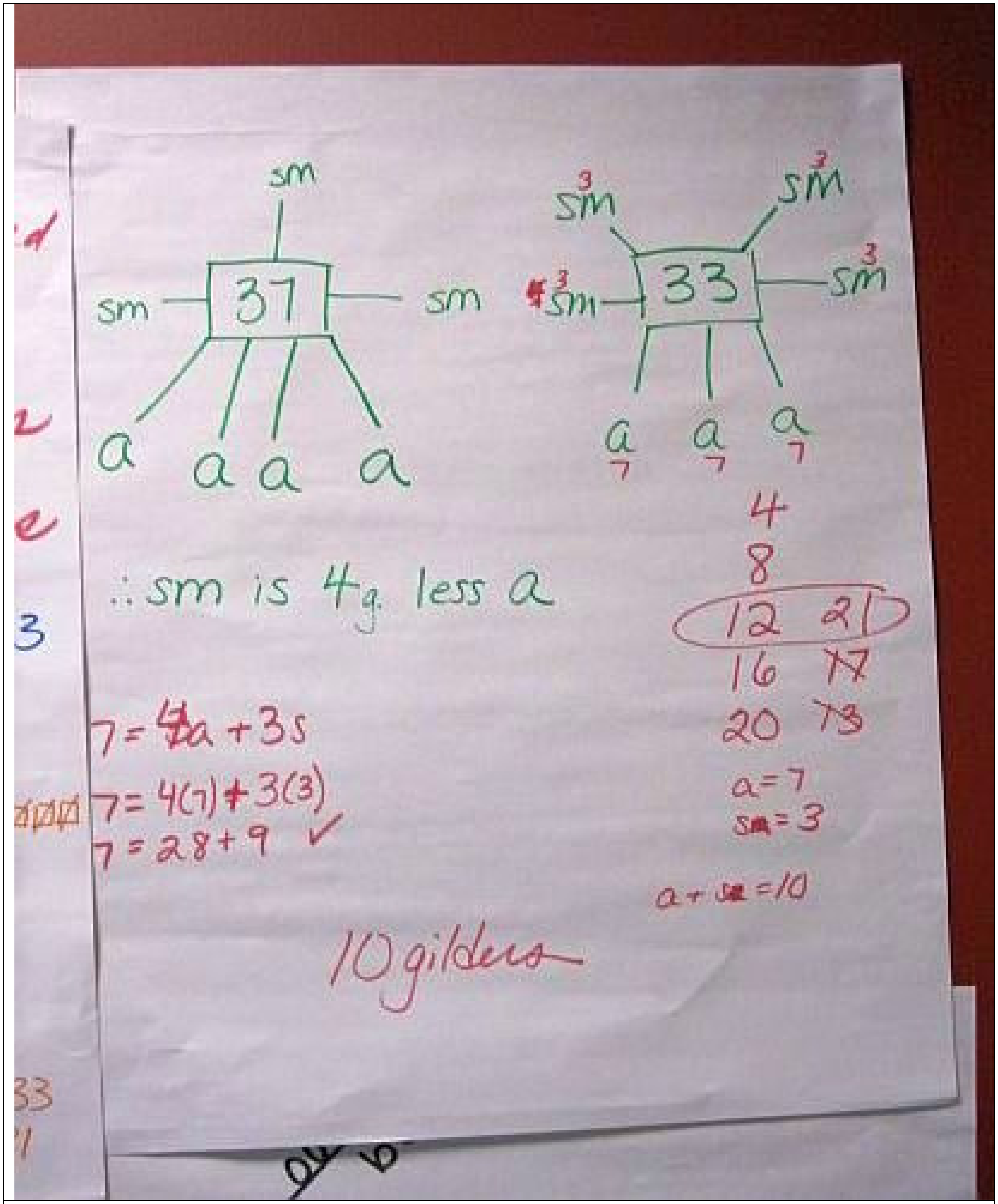

Figure 18
Room D

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