ECE297:11 Lecture 15

Elliptic Curve Cryptosystems

Elliptic Curve - General Equation

Set of solutions (x, y) to the equation

$$\mathbf{y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6}$$

where

$x, y \in K$

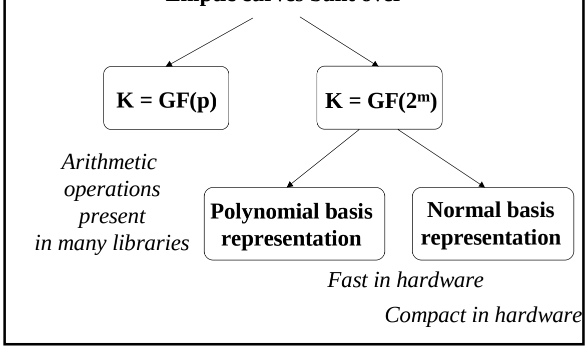
$a_1, a_2, a_3, a_4, a_5, a_6 \in K$

K is a field

Values of a_i limited by constraints specific to the field K

+ a special point called *the point at infinity* $\boldsymbol{O}$

Three Classes of Elliptic Curves



Elliptic curves built over

K = GF(p)

Arithmetic operations present in many libraries

K = GF(2^m)

Polynomial basis representation

Fast in hardware

Normal basis representation

Compact in hardware

Elliptic Curve over GF(p)

Set of solutions (x, y) to the equation

$$y^2 = x^3 + a\,x + b$$

where $x, y \in$ GF(p)

$a, b \in$ GF(p) $4a^3 + 27\,b^2 \not\equiv 0 \pmod{p}$

+ a special point called *the point at infinity* $\boldsymbol{O}$

Example: Elliptic curve $y^2 = x^3 + x + 1$ over GF(23)

(0, 1)
(0, 22)
(1, 7)
(1, 16)
(3, 10)
(3, 13)
(4, 0)
(5, 4)
(5, 19)
(6, 4)
(6, 19)
(7, 11)
(7, 12)
(9, 7)
(9, 16)
(11, 3)
(11, 20)
(12, 4)
(12, 19)
(13, 7)
(13, 16)
(17, 3)
(17, 20)
(18, 3)
(18, 20)
(19, 5)
(19, 18)
O

28 points

Generating a point of an elliptic curve (1)

1. Choose x
e.g., $x=3$

2. Compute $z = y^2 = x^3 + a\,x + b$
e.g., $z = 3^3 + 1 \cdot 3 + 1 \pmod{23} = 8$

3. If z = 0, then y=0 and there is only <u>one point</u>, (x,0), with the given *x* coordinate

Generating a point of an elliptic curve (2)

Otherwise

4. Verify whether there exists y such that $z = y^2$ (mod p) using Euler's criterion, i.e., check whether

$$z^{(p-1)/2} = 1 \pmod{p}$$

(if this is the case z is called a *quadratic residue mod p*)

e.g., $8^{(23-1)/2}$ (mod 23) = 8^{11} mod 23 =
= (8^8 mod 23)(8^2 mod 23)(8^1 mod 23) (mod 23) =
= 4 · 18 · 8 (mod 23) = 1

If Euler's criterion is not met (i.e., $z^{(p-1)/2} \neq 1$ (mod p)), then there is no point of the given elliptic curve with the given x coordinate

Generating a point of an elliptic curve (3)

Otherwise

5. If Euler's criterion is met, then there are two points with a given x coordinate

$$(x, y_1) \text{ and } (x, y_2)$$

If $p \equiv 3$ (mod 4) then

y_1 and y_2 can be computed from the equation

$$y_1 = +z^{(p+1)/4} \pmod{p}$$

$$y_2 = -z^{(p+1)/4} \pmod{p} \equiv p - z^{(p+1)/4} \pmod{p} = p - y_1$$

E.g., 23 ≡ 3 mod 4

$y_1 = 8^{(23+1)/4}$ mod 23 = 8^6 mod 23 = 13

$y_2 = -13 \equiv 23 - 13 = 10$

Addition of two points on the elliptic curve over GF(p) (1)

$$P = (x_1, y_1) \qquad Q = (x_2, y_2)$$

$$R = P + Q = (x_3, y_3)$$

Case 1:

$$P + O = O + P = P$$

Case 2:

$$x_2 = x_1 \text{ and } y_2 = -y_1$$

$$P + Q = O$$

$$Q = -P$$

Addition of two points on the elliptic curve over GF(p) (2)

Case 3:

$$x_3 = \lambda^2 - x_1 - x_2$$
$$y_3 = \lambda (x_1 - x_3) - y_1$$

where

Case 3a: if $P \neq Q$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} = (y_2 - y_1)(x_2 - x_1)^{-1}$$

Case 3b: if $P = Q$

$$\lambda = \frac{3x_1^2 + a}{2y_1} = (3x_1^2 + a)(2y_1)^{-1}$$

Example: Addition of points on the elliptic curve $y^2 = x^3 + x + 6$ over GF(11)

P = (2, 7)

2P = P + P = (2, 7) + (2, 7)

$\lambda = (3 \cdot 2^2 + 1)(2 \cdot 7)^{-1} \bmod 11 =$
$= 2 \cdot 3^{-1} \bmod 11 = 2 \cdot 4 \bmod 11 = 8$

$x_3 = 8^2 - 2 - 2 \bmod 11 = 9 - 2 - 2 \bmod 11 = 5$
$y_3 = 8 (2 - 5) - 7 \bmod 11 = 9 - 7 \bmod 11 = 2$

2P = (5, 2)

Example: Addition of points on the elliptic curve $y^2 = x^3 + x + 6$ over GF(11)

P = (2, 7) 2P = (5, 2)

3P = P + 2P = (2, 7) + (5, 2)

$\lambda = (2-7)(5-2)^{-1} \bmod 11 =$
$= 6 \cdot 3 \bmod 11 = 6 \cdot 4 \bmod 11 = 2$

$x_3 = 2^2 - 2 - 5 \bmod 11 = 4 - 2 - 5 \bmod 11 = 8$
$y_3 = 2 (2 - 8) - 7 \bmod 11 = 10 - 7 \bmod 11 = 3$

3P = (8, 3)

Scalar multiples of P

P = (2, 7)
2P = (5, 2)
3P = (8, 3)
4P = (10, 2)
5P = (3, 6)
6P = (7, 9)
7P = (7, 2)
8P = (3, 5)
9P = (10, 9)
10P = (8, 8)
11P = (5, 9)
12P = (2, 4)
13P = O

Number of points on the curve = 13
P is a generator of the group of points on the elliptic curve

Number of points on the curve #E(GF(p))
= order of an elliptic curve
= cardinality of an elliptic curve

Hasse's Theorem

$$p+1-2\sqrt{p} \le \#E(GF(p)) \le p+1+2\sqrt{p}$$

e.g.,

order of a curve over GF(11)

$$11+1-2\sqrt{11} \le \#E(GF(11)) \le 11+1+2\sqrt{11}$$

$$5.37 \le \#E(GF(11)) \le 18.63$$

order of the curve $y^2 = x^3 + x + 6$ over GF(11) = 13

Number of points on the curve #E(GF(p))

Exact number #E(GF(p)) can be computed using
Schoof's algorithm

Complexity: $(\log p)^8$

To prevent the Pohlig-Hellman method of computing elliptic curve discrete logarithm:
#E(GF(p)) must have a large prime divisor

"Large" currently means ~ 10^{40}

Exponentiation: $y = a^e \bmod n$

Right-to-left binary exponentiation

Left-to-right binary exponentiation

$e = (e_{L-1}, e_{L-2}, \ldots, e_1, e_0)_2$

```
$y = 1$;
$s = a$;
for $i$=0 to L-1
  {
    if ($e_i$ == 1)
      $y = y \cdot s \bmod n$;
    $s = s^2 \bmod n$;
  }
```

```
$y = 1$;
for $i$=L-1 downto 0
  {
    $y = y^2 \bmod n$;
    if ($e_i$ == 1)
      $y = y \cdot a \bmod n$;
  }
```

Scalar Multiplication: $Y = k \cdot P$

Right-to-left binary scalar multiplication

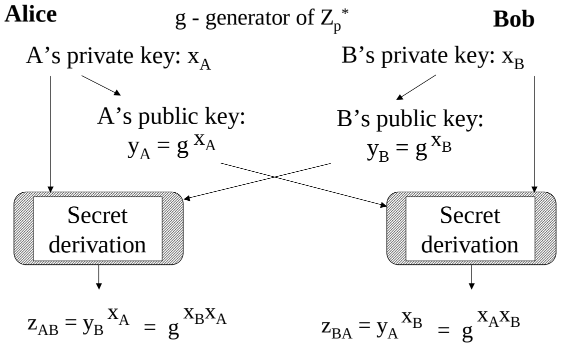
Left-to-right binary scalar multiplication

$k = (k_{L-1}, k_{L-2}, \ldots, k_1, k_0)_2$

```
$Y = O$;
$S = P$;
for $i$=0 to L-1
  {
    if ($k_i$ == 1)
      $Y = Y + S$;
    $S = 2S$ ;
  }
```

```
$Y = O$;
for $i$=L-1 downto 0
  {
    $Y = 2Y$ ;
    if ($k_i$ == 1)
      $Y = Y + P$;
  }
```

Diffie-Hellman



Alice

g - generator of Z_p^*

Bob

A's private key: x_A

B's private key: x_B

A's public key: $y_A = g^{x_A}$

B's public key: $y_B = g^{x_B}$

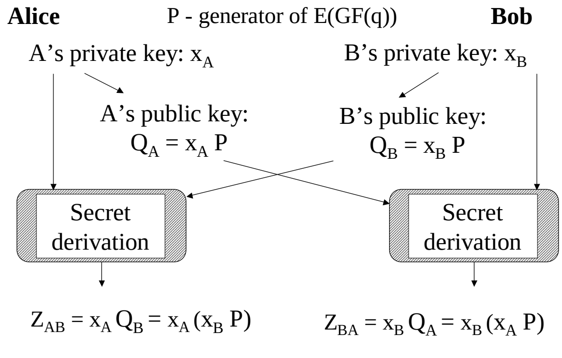
Secret derivation

Secret derivation

$z_{AB} = y_B^{\ x_A} = g^{x_B x_A}$

$z_{BA} = y_A^{\ x_B} = g^{x_A x_B}$

Elliptic Curve Diffie-Hellman



Alice P - generator of E(GF(q)) **Bob**

A's private key: x_A

B's private key: x_B

A's public key: $Q_A = x_A P$

B's public key: $Q_B = x_B P$

Secret derivation

Secret derivation

$Z_{AB} = x_A Q_B = x_A (x_B P)$

$Z_{BA} = x_B Q_A = x_B (x_A P)$

Digital Signature Algorithm

System parameters

May be shared by a group of users or belong to a single user; known to everybody

q - 160-bit prime
p - L-bit prime, such that q | p-1
where L = 1024 + 64·k

g = $h^{(p-1)/q}$ mod p where $1 < h < p-1$, such that g>1

From Fermat's theorem
g^q mod p = h^{p-1} mod p = 1
g - generator of the cyclic group of order q in Zp*

Elliptic Curve Digital Signature Algorithm ECDSA

System parameters

May be shared by a group of users or belong to a single user; known to everybody

E - elliptic curve over GF(*p*) or GF(2^m)

P - point of order ***q*** on the elliptic curve E

Digital Signature Algorithm

Public and private key

Private key

x - arbitrary 160 bit number $0 < x < q$

Public key

$y = g^x \bmod p$ $0 < y < p$

L - bit number

Elliptic Curve Digital Signature Algorithm

Public and private key

Private key

x - arbitrary number $0 < x < q$

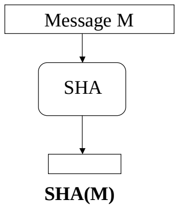
Public key

$Y = x P$

DSA: Signature generation

1. Choose random *message private key* $1 < \boldsymbol{k} < q$ (secret, different for each message)
2. Compute *message public key* $\boldsymbol{r} = (g^k \bmod p) \bmod q$

3. Compute *hash value*



Message M

SHA

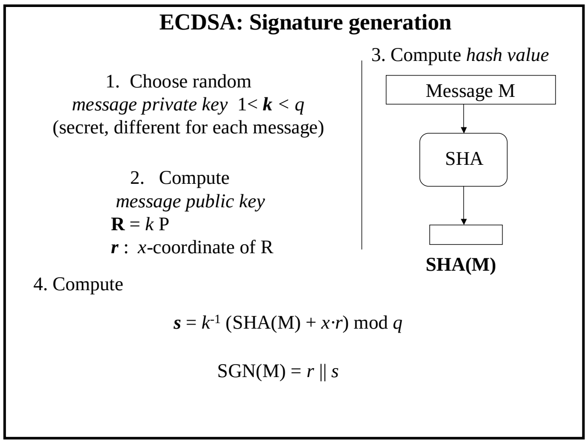
SHA(M)

4. Compute

$$\boldsymbol{s} = k^{-1}\,(\text{SHA(M)} + x \cdot r) \bmod q$$

$$\text{SGN(M)} = \underbrace{r}_{\text{160 bit}} \,||\, \underbrace{s}_{\text{160 bit}}$$

40 bytes



ECDSA: Signature generation

1. Choose random *message private key* $1 < \mathbf{k} < q$ (secret, different for each message)

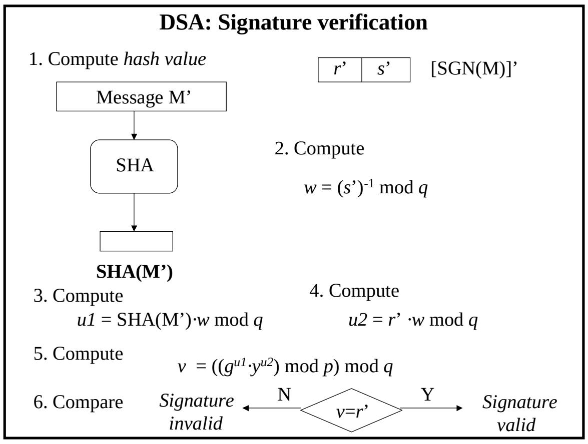
2. Compute *message public key*
R = k P
$\boldsymbol{r}$: x-coordinate of R

3. Compute *hash value*

4. Compute

$$\boldsymbol{s} = k^{-1}\,(\mathrm{SHA(M)} + x \cdot r) \bmod q$$

$$\mathrm{SGN(M)} = r \,\|\, s$$



DSA: Signature verification

1. Compute *hash value*

2. Compute

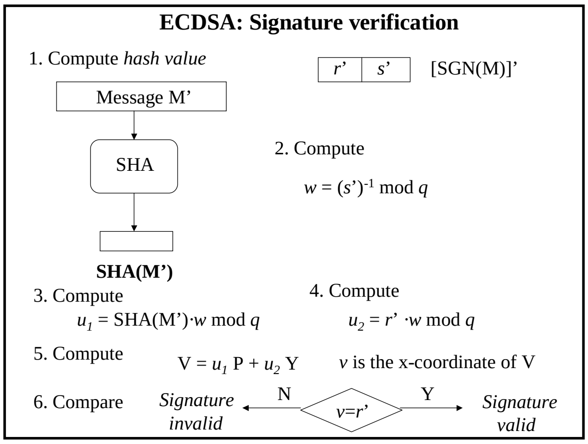
$w = (s')^{-1} \bmod q$

3. Compute
$u1 = \mathrm{SHA(M')} \cdot w \bmod q$

4. Compute
$u2 = r' \cdot w \bmod q$

5. Compute
$v = ((g^{u1} \cdot y^{u2}) \bmod p) \bmod q$

6. Compare



ECDSA: Signature verification

1. Compute *hash value*

2. Compute

$w = (s')^{-1} \bmod q$

3. Compute
$u_1 = \mathrm{SHA(M')} \cdot w \bmod q$

4. Compute
$u_2 = r' \cdot w \bmod q$

5. Compute
$V = u_1\,P + u_2\,Y$ $\quad$ v is the x-coordinate of V

6. Compare

El-Gamal Encryption

System parameters

May be shared by a group of users or belong to a single user; known to everybody

p - prime

g - generator of the group Zp*

Elliptic Curve El-Gamal Encryption

System parameters

May be shared by a group of users or belong to a single user; known to everybody

E - elliptic curve over GF(*p*) or GF(2^m)

P - generator of the group of points on the elliptic curve

El-Gamal Encryption

Public and private key

Private key

x - arbitrary number $1 \le x \le p-2$

Public key

$y = g^x \bmod p$ $0 < y < p$

Elliptic Curve El-Gamal Encryption

Public and private key

Private key

x - arbitrary number $1 \le x \le \#E(GF(q))-1$

Public key

$Y = x\,P$

El-Gamal: Encryption

1. Choose random
message private key $1 \le \mathbf{k} \le p\text{-}2$,
relatively prime with p-1
(secret, different for each message)

2. Compute
message public key
$\mathbf{r} = g^k \bmod p$

3. Compute

$\mathbf{c} = y^k \cdot M \bmod p$

$C(M) = \mathbf{r} \,||\, \mathbf{c}$

Elliptic Curve El-Gamal: Encryption

1. Choose random
message private key $1 \le \mathbf{k} \le \#E(GF(q))\text{-}1$,
(secret, different for each message)

2. Compute
message public key
$\mathbf{R} = k\,P$

3. Compute

$\mathbf{C} = k\,Y + M \bmod p$

$C(m) = \mathbf{R} \,||\, \mathbf{C}$

3. Compute

$\mathbf{M} = (m, n)$

***m* - message**
n - y-coordinate
corresponding
to the x-coordinate *m*

El-Gamal: Decryption

r	*c*

C(*M*)

$M = c \cdot (r^x)^{-1} \bmod p$

Justification:

$c \cdot (r^x)^{-1} \bmod p = y^k \cdot M \cdot ((g^k)^x)^{-1} = y^k \cdot M \cdot ((g^x)^k)^{-1} =$
$= y^k \cdot M \cdot (y^k)^{-1} = M$

Elliptic Curve El-Gamal: Decryption

R	*C*

C(*m*)

$\mathrm{M} = \mathrm{C} - x\,\mathrm{R}$

m: *x*-coordinate of M

Justification:

$\mathrm{C} - x\,\mathrm{R} = (k\,\mathrm{Y} + \mathrm{M}) - x\,\mathrm{R} = (k\,\mathrm{Y} + \mathrm{M}) - x\,k\,\mathrm{P} =$
$= (k\,\mathrm{Y} + \mathrm{M}) - k\,(x\,\mathrm{P}) = k\,\mathrm{Y} + \mathrm{M} - k\,\mathrm{Y} = \mathrm{M}$

Menezes-Vanstone Elliptic Curve Cryptosystem

System parameters

May be shared by a group of users or belong to a single user;
known to everybody

E - elliptic curve over GF(*p*) or GF(2^m)

P - generator of the group of points
on the elliptic curve

Menezes-Vanstone Elliptic Curve Cryptosystem

Public and private key

Private key

x - arbitrary number $1 \le x \le \#E(GF(q))-1$

Public key

$Y = x\,P$

Menezes-Vanstone Cryptosystem: Encryption

1. Choose random
message private key $1 \le \boldsymbol{k} \le \#E(GF(q))-1$,
(secret, different for each message)

2. Compute
message public key
R = k P

3. Form message block:
(m_1, m_2)

4. Compute

$\boldsymbol{C} = k\,Y = (c_1, c_2)$

5. Compute

$y_1 = c_1 m_1$
$y_2 = c_2 m_2$

$C(m_1, m_2) = \boldsymbol{R} \,||\, \boldsymbol{y_1, y_2}$

Menezes Vanstone Cryptosystem : Decryption

R	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$

$C(m_1, m_2)$

$C = x\,R = (c_1, c_2)$

$m_1 = c_1^{-1}\,y_1$
$m_2 = c_2^{-1}\,y_2$

Justification:

$x\,R = x\,k\,P = k\,(x\,P) = k\,Y = C$