2.5 General Variable Partitions

This section addresses the general partition case. Let $\mathbb{P} = (\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_M)$ be a partition of $\vec{x}$, where $\vec{y}_i$ is the subset of variables at site i, ($\mid \vec{y}_i \mid = n_i$), and $\vec{x}$ is the vector of all variables in our problem.

The problem in the general case is that, for a safe decomposition, a constraint C_i at site i may be characterized by an unbounded set of atomic linear constraints; thus the size of a parametric description (using coefficients of those constraints) is unbounded. To overcome this problem, we reduce the search space to the set of what we call *compact split* decompositions, for which we prove that: (1) there does exist a parametric description of bounded size and (2) the optimum of the objective function among all safe decompositions *can always be found* in the subspace of split decompositions.

2.5.1 Split Decompositions

Definition 12. *Let $\Omega = A\vec{x} \leq \vec{b}$ be a constraint on $\vec{x}$, and $\mathbb{P} = (\vec{y}_1, \ldots, \vec{y}_M)$ be a variable partition of $\vec{x}$. A* split *of Ω, denoted by $D(\vec{r}_1, \ldots, \vec{r}_M)$, is a tuple $(A_1\vec{y}_1 \leq \vec{r}_1, \ldots, A_M\vec{y}_M \leq \vec{r}_M)$ of constraints, where A_i, $1 \leq i \leq M$, are those columns of A associated with $\vec{y}_i$. We say that a split $D(\vec{r}_1, \ldots, \vec{r}_M)$ is safe (respectively minimally-constrained) if it is a safe (respectively minimally-constrained) decomposition of Ω. For a subset θ of sites, say $\{k+1, \ldots, M\}$, a (partial) θ-split of Ω, denoted by $D(\vec{r}_{k+1}, \ldots, \vec{r}_M)$, is a tuple $(A_{k+1}\vec{y}_{k+1} \leq \vec{r}_{k+1}, \ldots, A_M\vec{y}_M \leq \vec{r}_M)$ of constraints.*

Recall that, by Proposition 4, $D(\vec{r}_1, \ldots, \vec{r}_M)$ is satisfiable if and only if for all i, $1 \leq i \leq M$, $A_i\vec{y}_i \leq \vec{r}_i$ is satisfiable.

For our classification we introduce the notion of *tight form* for a system $A\vec{x} \leq \vec{b}$, which states, intuitively, that the values of $\vec{b}$ are tight. This is formalized as follows.

Definition 13. *We say that a constraint $A\vec{x} \leq \vec{b}$ is* tight*, if there does not exist $\vec{b}'$, such that $\vec{b}' \leq \vec{b}$, $\vec{b}' \neq \vec{b}$, and $A\vec{x} \leq \vec{b}$ is equivalent to $A\vec{x} \leq \vec{b}'$. We say that a split $D(\vec{r}_1, \ldots, \vec{r}_M)$ is tight if every satisfiable constraint $A_i\vec{y}_i \leq \vec{r}_i$ in it ($1 \leq i \leq M$) is tight.*

Claim 1. *For any satisfiable system $A\vec{x} \leq \vec{b}$ (respectively safe split) there exists an equivalent system (respectively split) that is tight. Furthermore, every tight constraint $A\vec{x} \leq \vec{b}$ is satisfiable.*

Definition 14. *We say that a split $D(\vec{r}_1, \ldots, \vec{r}_M)$ of Ω is* compact *if*

$$\sum_{i=1}^{M} \vec{r}_i \leq \vec{b}$$

Lemma 1 (Split Properties). *Let $\Omega = A\vec{x} \leq \vec{b}$. Then:*

1. *Every compact split is safe.*

2. *If $D(\vec{r}_1, \ldots, \vec{r}_M)$ is a tight split, it is compact iff it is safe.*

3. *For every safe decomposition $\mathbb{C}$ of Ω, there exists a minimally-constrained safe split $D(\vec{r}_1, \ldots, \vec{r}_M)$ of Ω, that subsumes $\mathbb{C}$, i.e., $\mathbb{C} \models D(\vec{r}_1, \ldots, \vec{r}_M)$.*

4. Every minimally-constrained safe decomposition of Ω is equivalent to (1) a minimally - constrained safe split of Ω and to (2) a compact split of Ω.

Proof. (1) Let $\vec{x}^0 = (\vec{y}_1^0, \ldots, \vec{y}_M^0)$ be an arbitrary point that satisfies $D(\vec{r}_1, \ldots, \vec{r}_M)$, i.e., $A_i\vec{y}_i^0 \leq r_i$, for all i, $1 \leq i \leq M$. Then,

$$\sum_{i=1}^{M} A_i\vec{y}_i^0 \leq \sum_{i=1}^{M} \vec{r}_i \leq \vec{b}$$

Finally, since $\sum_{i=1}^{M} A_i\vec{y}_i^0 = A\vec{x}^0$, we get that $A\vec{x}^0 \leq \vec{b}$, i.e., $\vec{x}^0$ satisfy Ω.

(2) The IF-direction is subsumed by (1). For the ONLY-IF-direction, suppose $\sum_{i=1}^{M} \vec{r}_i \leq \vec{b}$ is not true. Then, there exists a row j, such that $\sum_{i=1}^{M} r_{ij} > b_j$. Now, we select $\vec{x}^0 = \{\vec{y}_1^0, \vec{y}_2^0, \ldots, \vec{y}_M^0\}$, as follows. For each $i, 1 \leq i \leq M$, we take $\vec{y}_i^0$ to be the point that achieves $Max\ \vec{A}_{ij}\vec{y}_i$, subject to $A\vec{y}_i \leq \vec{r}_i$, with $\vec{A}_{ij}$ is the jth row of matrix A. Because the system $A\vec{y}_i \leq \vec{r}_i$ is in reduced form, $\vec{A}_{ij}\vec{y}_i^0 = r_{ij}$. Thus, $\vec{x}^0$ satisfies $D(\vec{r}_1, \ldots, \vec{r}_M)$ but not $\Omega = A\vec{x} \leq \vec{b}$, because

$$\vec{A}_{ij}\vec{x}^0 = \sum_{i=1}^{M} \vec{A}_{ij}\vec{y}_i^0 = \sum_{i=1}^{M} r_{ij} > b_j$$

This complete the ONLY-IF direction of (2).

(3) We first prove that:

Claim 2. *For every safe decomposition* $\mathbb{C}$ *of* Ω*, there exists a safe split* $D(\vec{r}_1, \ldots, \vec{r}_M)$ *that subsumes* $\mathbb{C}$*.*

Let $\mathbb{C} = (C_1, \ldots, C_M)$ be a safe decomposition of Ω. We want to construct $D(\vec{r}_1, \ldots, \vec{r}_M)$. Consider first the case when $\mathbb{C}$ is satisfiable. For every $i, 1 \leq i \leq m$, where m is the number of rows in matrix A, we construct $\vec{r}_i = (r_{i1}, \ldots, r_{im})$ as follows: for every $j, 1 \leq j \leq m$, we take r_{ij} to be the minimal value such that

$$C_i \models \vec{A}_{ij}\vec{y}_i \leq r_{ij}$$

Then, $C_i \models A_i\vec{y}_i \leq \vec{r}_i$, and therefore $\mathbb{C} \models D(\vec{r}_1, \ldots, \vec{r}_M)$. Therefore, by (1) of this Theorem, to prove that $D(\vec{r}_1, \ldots, \vec{r}_M)$ is a safe decomposition, it is sufficient to prove that $D(\vec{r}_1, \ldots, \vec{r}_M)$ is a safe split, i.e., $\sum_{i=1}^{M} \vec{r}_i \leq \vec{b}$. We prove that for every row $j, 1 \leq j \leq m$, i.e., $\sum_{i=1}^{M} r_{ij} \leq b_j$.

Selecting $\vec{x}^{(j)} = (\vec{y}_1^{(j)}, \ldots, \vec{y}_M^{(j)})$ is done as follows: $\vec{y}_i^{(j)}$ is the value of $\vec{y}_i$ that maximizes the function $\vec{A}_{ij}\vec{y}_i$, subject to $A_i\vec{y}_i \leq \vec{r}_i$. Because $D(\vec{r}_1, \ldots, \vec{r}_M)$ is constructed so that it is in a reduced form, $\vec{A}_{ij}\vec{y}_i^{(j)} = r_{ij}$. Then, for every $j, 1 \leq j \leq m$

$$\sum_{i=1}^{M} d_{ij} = \sum_{i=1}^{M} \vec{A}_{ij}\vec{y}_i^{(j)} = \vec{A}_j\vec{x}^{(j)} \leq b_j$$

This completes the proof of Claim 2 for the case when $\mathbb{C}$ is satisfiable.

If $\mathbb{C}$ is not satisfiable, consider $\mathbb{C}' = (\vec{x} = \vec{x}^0)$, where $\vec{x}^0$ is an arbitrary point that

satisfies Ω. Clearly,$\mathbb{C}'$ is satisfiable. By what has been proved, we can construct a safe decomposition $D(\vec{r}_1, \ldots, \vec{r}_M)$ of Ω such that:

$$\mathbb{C}' \models D(\vec{r}_1, \ldots, \vec{r}_M)$$

and, since $\mathbb{C} \models \mathbb{C}'$, it follows that $\mathbb{C} \models D(\vec{r}_1, \ldots, \vec{r}_M)$. This complete the proof of Claim 2.

Claim 3. *Every minimally-constrained safe decomposition $\mathbb{C}'$ is equivalent to a minimally - constrained safe split $D(\vec{r}_1, \ldots, \vec{r}_M)$.*

Indeed, by Claim 2, there exists a safe split $D(\vec{r}_1, \ldots, \vec{r}_M)$ such that $\mathbb{C}' \models D(\vec{r}_1, \ldots, \vec{r}_M)$. Then, because $\mathbb{C}'$ is minimally-constrained, $D(\vec{r}_1, \ldots, \vec{r}_M)$ must be equivalent to $\mathbb{C}'$. Therefore, $D(\vec{r}_1, \ldots, \vec{r}_M)$ is a minimally-constrained safe split.

Now, to prove (3) given a safe decomposition $\mathbb{C}$, we construct a minimally-constrained safe decomposition $\mathbb{C}'$, such that $\mathbb{C} \models \mathbb{C}'$. Then, by Claim 3, there exists a minimally-constrained safe split $D(\vec{r}_1, \ldots, \vec{r}_M)$ that is equivalent to $\mathbb{C}'$. Therefore, $D(\vec{r}_1, \ldots, \vec{r}_M)$ is a minimally-constrained safe split that subsumes $\mathbb{C}$. This completes the proof of (3).

(4) It is essentially Claim 3 that has been proved in (3). $\square$

The next theorem shows that the maximum of a monotonic function in the space of safe decompositions can always be found in the subspace of compact splits.

Theorem 3. *Let $\Omega = A\vec{x} \leq \vec{b}$ be a satisfiable global constraint, f be a monotonic function from the set of all safe decompositions of Ω to $\mathbb{R}$, $\mathbb{P} = (\vec{y}_1, \ldots, \vec{y}_M)$ be a variable partition of $\vec{x}$, and $\vec{x}^0 = (\vec{y}_1^0, \ldots, \vec{y}_M^0)$ be an instance of $\vec{x}$. Let $\mathbb{S}$ and $\mathbb{SS}$ be the sets of all safe decompositions and all compact splits of Ω, respectively, and let $\mathbb{S}_{\vec{x}^0}$ and $\mathbb{SS}_{\vec{x}^0}$ be the sets $\mathbb{S}$ and $\mathbb{SS}$ restricted to decompositions that satisfy $\vec{x}^0$. Then,*

1. $max\ f(s)\ s.t.\ s \in \mathbb{S} = max\ f(s)\ s.t.\ s \in \mathbb{SS}$

2. $max\ f(s)\ s.t.\ s \in \mathbb{S}_{\vec{x}^0} = max\ f(s)\ s.t.\ s \in \mathbb{SS}_{\vec{x}^0}$

Proof. (1) The proof follows from the fact that $\mathbb{SS} \subseteq \mathbb{S}$. Thus, by Lemma 1, $\mathbb{SS}$ is a *monotonic cover* of $\mathbb{S}$. Then, using Proposition 5, both problems yield the same maximum.

(2) The proof follows directly from 1), because 2) is a particular case of 1). This completes the proof. □

Following Theorem 3, from now on we only consider compact splits. Vectors $(\vec{r}_1, \ldots, \vec{r}_M)$ can be considered as resources assigned to sites, because they represent how much of vector $\vec{b}$ is distributed to each site. The following subsection presents a parametric resource characterization of splits and a parametric formulation of the optimization problem in terms of resources.

2.5.2 Resource Characterization

This subsection characterizes (compact) splits in terms of resources. This resource-based characterization supports a concurrent constraint (re-) decompositions.

In particular, we formulate the properties of *compactness*, *local consistency (lc)*, *partial constraint preservation (pcp)*, and *resource bound partition (rp)* for splits in terms of resources. Then, the optimization problem is formulated in terms of such a characterization. First, we introduce the concept of *resources* of (compact) splits.

Definition 15 (Resource Parameters). *Let $\Omega = A\vec{x} \leq \vec{b}$ be a satisfiable global constraint, $D(\vec{r}_1, \ldots, \vec{r}_M)$ be a tight compact split of Ω, $\vec{x}^0 = (\vec{y}_1^0, \ldots, \vec{y}_M^0)$ be an instance of $\vec{x}$, and θ be a subset $\{k+1, \ldots, M\}$ of sites $\{1, \ldots, M\}$. Then, we say that:*

1. *$\vec{b}$ is the* global upper bound *of resources in Ω.*

2. *$\vec{r}_i$ is the* resource *assigned to site i, $1 \leq i \leq M$.*

3. *$\vec{r} = \sum_{i=1}^{M} \vec{r}_i$ is the* global resource.

4. *$\vec{\delta} = \vec{b} - \vec{r}$ is the* global passive slack *of Ω w.r.t. $D(\vec{r}_1, \ldots, \vec{r}_M)$.*

5. *$(\vec{\delta}_1, \ldots, \vec{\delta}_M)$ such that $\vec{\delta}_i \geq 0$, $1 \leq i \leq M$ and $\sum_{i=1}^{M} \vec{\delta}_i = \vec{\delta}$, is a partition of $\vec{\delta}$. Each $\vec{\delta}_i$, $1 \leq i \leq M$, is called the passive slack at site i.*

6. *$\vec{ur}_i = \vec{r}_i + \vec{\delta}_i$ is the* upper resource bound *at site i, $(1 \leq i \leq M)$.*

7. *Given an instance* $\vec{y}_i^0$ *at site* i, $\vec{lr}_i = A_i\vec{y}_i^0$, $1 \leq i \leq M$, *is the* lower resource bound *at site* i *w.r.t.* $\vec{y}_i^0$.

8. *Given an instance* $\vec{y}_i^0$ *at site* i, $\vec{\Delta}_i = \vec{r}_i - \vec{lr}_i$, $1 \leq i \leq M$, *is the* active slack *at site* i *w.r.t.* $\vec{y}_i^0$.

Finally, we define all the parameters for θ *by* $\vec{r}_\theta = \sum_{i\in\theta} \vec{r}_i$, $\vec{ur}_\theta = \sum_{i\in\theta} \vec{ur}_i$, $\vec{lr}_\theta = \sum_{i\in\theta} \vec{lr}_i$, $\vec{\delta}_\theta = \sum_{i\in\theta} \vec{\delta}_i$, *and* $\vec{\Delta}_\theta = \sum_{i\in\theta} \vec{\Delta}_i$.

The above resource parameters are shown in Figure 2.2. In this figure, each resource $\vec{r}_i$ is bounded between its lower and upper bound ($\vec{lr}_i$ and $\vec{ur}_i$), the difference between upper bound ($\vec{ur}_i$) and the resource is the passive slack $\vec{\delta}_i$, and the difference between the resource ($\vec{r}_i$) and its lower bound ($\vec{lr}_i$) is the active slack ($\vec{\Delta}_i$).

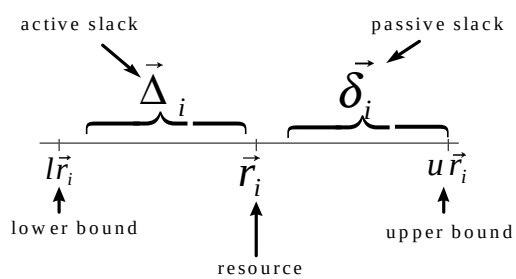

Figure 2.2: Resource Representation at Site i

The following proposition characterizes the split properties of *compactness*, *local consistency (lc)*, *partial constraint preservation (pcp)*, and *resource bound partition (rp)* in terms of the resource parameters.

Proposition 8 (Parametric Feasible Properties). *Let* $\Omega = A\vec{x} \leq \vec{b}$ *be a global satisfiable constraint,* $\mathbb{P} = (\vec{y}_1, \ldots, \vec{y}_M)$ *be a variable partition of* $\vec{x}$, $\vec{x}^0 = (\vec{y}_1^0, \ldots, \vec{y}_M^0)$

be an instance of $\vec{x}$*,* θ *be a subset of sites, say* $\theta = \{k+1, \ldots, M\}$*,* $\bar{\theta}$ *be the set* $\{1, \ldots, k\}$*, and* $D(\vec{r}_1^0, \ldots, \vec{r}_k^0)$ *be a (partial)* $\bar{\theta}$*-split that satisfies* $(\vec{y}_1^0, \ldots, \vec{y}_k^0)$*. Then, for any split* $D(\vec{r}_1, \ldots, \vec{r}_M)$

1. $D(\vec{r}_1, \ldots, \vec{r}_M)$ *is compact iff the global resource* $\vec{r}$ *is bounded by the global upper bound* $\vec{b}$*, i.e.,* $\vec{r} \leq \vec{b}$*. We denote this condition by* $\Phi_{compact}(\vec{r}_1, \ldots, \vec{r}_M)$*.*

2. $D(\vec{r}_1, \ldots, \vec{r}_M)$ *satisfies local consistency w.r.t.* $\vec{x}^0$ *iff the resource* $\vec{r}_i$ *assigned to site* i *is bounded from below by its lower bound* $\vec{lr}_i$*, i.e., for every site* i*,* $1 \leq i \leq M$*,* $\vec{lr}_i \leq \vec{r}_i$*. We denote this condition by* $\Phi_{lc}(\vec{r}_1, \ldots, \vec{r}_M)$*.*

3. $D(\vec{r}_1, \ldots, \vec{r}_M)$ *satisfies partial constraint preservation w.r.t.* $\bar{\theta}$*-split* $D(\vec{r}_1^0, \ldots, \vec{r}_k^0)$ *iff the resources at each site outside* θ *are fixed, i.e.,* $r_1 = r_1^0, \ldots, r_k = r_k^0$*. We denote this condition by* $\Phi_{pcp}(\vec{r}_1, \ldots, \vec{r}_M)$*.*

We will also denote by $\Phi_{rp,\vec{B}_\theta}(\vec{r}_1, \ldots, \vec{r}_M)$ *the condition stating that* $D(\vec{r}_1, \ldots, \vec{r}_M)$ *satisfies resource partition w.r.t. a resource bound partition* $\vec{B}_\theta$*.*

Proof. 1) follows directly from Lemma 1 part 1, 2) follows from the definition of local consistency, i.e., for a given instance $\vec{y}_i^0$, it satisfies its local constraint iff $A_i\vec{y}_i^0 \leq \vec{r}_i$, and 3) follows directly from partial constraint preservation definition. □

In the following, we denote by $\mathbb{SS}_{compact}$, $\mathbb{SS}_{lc}$, $\mathbb{SS}_{pcp}$ and $\mathbb{SS}_{rp}$ the set of splits satisfying compactness, local consistency w.r.t. $\vec{x}^0$, partial constraint preservation w.r.t. a θ-split $D(\vec{r}_1, \ldots, \vec{r}_k)$, and resource bound partition w.r.t. θ and bound B_θ respectively. We will use Pr to denote a subset of the set of properties $\{compact, lc, pcp, rp\}$.

Finally, set $\mathbb{S}_{Pr}$ will denote the set of all splits that satisfy the properties Pr, i.e., $\mathbb{SS}_{Pr} = \cap_{p \in Pr} \mathbb{SS}_p$, and $\Phi_{Pr}(\vec{r}_1, \ldots \vec{r}_M)$ will be the conjunction of the corresponding conditions, i.e., $\Phi_{Pr}(\vec{r}_1, \ldots \vec{r}_M) = \wedge_{p \in Pr} \Phi_p(\vec{r}_1, \ldots \vec{r}_M)$. We can present the optimization problem in terms of resource characterization.

Theorem 4 (Resource Optimization). *Let $\Omega = A\vec{x} \leq \vec{b}$ be a satisfiable global constraint, f be a monotonic function from the set of all safe decompositions to $\mathbb{R}$, $\mathbb{P} = (\vec{y}_1, \ldots, \vec{y}_M)$ a variable partition of $\vec{x}$, $\vec{x}^0 = (\vec{y}_1^0, \ldots, \vec{y}_M^0)$ be an instance of $\vec{x}$, and Pr be the subset of properties $\{compactness, lc, pcp, rp\}$ that must contain compactness or resource bound partition. Then, solving the optimization problem*

$$max\ f(s)$$

$$s.t.\ s \in \mathbb{SS}_{Pr}$$

is equivalent to solving the parametric problem [10]

$$max\ f(D(\vec{r}_1, \ldots, \vec{r}_M))$$

$$s.t.\ \Phi_{Pr}(\vec{r}_1, \ldots, \vec{r}_M)$$

Proof. The proof follows from Lemma 1, Proposition 8, and the fact that resource bound partition is a stronger property than compactness as follows:

(1) The *compactness* part. By Proposition 8, the set $\mathbb{SS}_{compactness}$ is characterized

[10] we will sometimes write $f(D(\vec{r}_1, \ldots, \vec{r}_M))$ as $f(\vec{r}_1, \ldots, \vec{r}_M)$.

by condition $\Phi_{compactness}(\vec{r}_1, \ldots, \vec{r}_M)$. Then, both problems yield the same maximum.

The other Pr cases follow directly from Proposition 8. This completes the proof. □

2.5.3 Concurrent Split Decompositions

The resource characterization of the previous subsection assumes that information from all sites is used. However, in order to support distributed and autonomous protocols we would like to make constraint decompositions and re-decompositions involving only a (small) subset of sites, say $\theta = \{(k+1), \ldots, M\}$ of sites $\{1, \ldots, M\}$. To do that a formulation of the decomposition problem can only involve or affect information that is stored in sites θ.

Definition 16. *Let $\Omega = A\vec{x} \leq \vec{b}$ be a global constraint, $D(\vec{r}_1, \ldots, \vec{r}_M)$ be a compact split of Ω, $(\vec{\delta}_1, \ldots, \vec{\delta}_M)$ be a partition of the global passive slack $\vec{\delta}$, $\vec{x}^0 = (\vec{y}_1^0, \ldots, \vec{y}_M^0)$ be a database instance, θ is a subset, say $\{k+1, \ldots, M\}$, of sites. A* resource distribution *is a tuple (of triples) $((\vec{lr}_1, \vec{r}_1, \vec{ur}_1), \ldots, (\vec{lr}_M, \vec{r}_M, \vec{ur}_M))$ [11]; a θ-resource distribution is $((\vec{lr}_{k+1}, \vec{r}_{k+1}, \vec{ur}_{k+1}), \ldots, (\vec{lr}_M, \vec{r}_M, \vec{ur}_M))$. We say that the resource distribution is* permissible *if*

$$\sum_{i=1}^{M} \vec{ur}_i = \vec{b} \text{ and } \vec{lr}_i \leq \vec{r}_i \leq \vec{ur}_i, \text{for every } 1 \leq i \leq M$$

Given a resource bound partition $\vec{B}_\theta$, we say that the θ-resource distribution $((\vec{lr}_{k+1},$

[11] Note $(\vec{lr}_i, \vec{r}_i, \vec{ur}_i)$'s are defined in Definition 15.

$\vec{r}_{k+1}, \vec{ur}_{k+1}), \ldots, (\vec{lr}_M, \vec{r}_M, \vec{ur}_M))$ *is permissible w.r.t.* $\vec{B}_\theta$ *if*

$$\sum_{i \in \theta} \vec{ur}_i = \vec{B}_\theta \text{ and } \vec{lr}_i \leq \vec{r}_i \leq \vec{ur}_i, \text{for every } i \in \theta$$

Note that if θ is the set of all sites, the resource distribution permissibility is equivalent to θ resource distribution permissibility.

The following proposition motivates the notion of permissible distribution and provides a local criterion for a subset θ of sites to decide whether a feasible resource distribution exists, after database instances have been updated (i.e., lower bounds).

Proposition 9 (θ-Resource Distribution Feasibility). *Let θ be a subset, say $\{k+1, \ldots, M\}$, of sites $\{1, \ldots, M\}$, and $\bar{\theta}$ be its complement, $\vec{B}_\theta$ be a resource bound partition. Then,*

1. *Given a database instance $(\vec{y}^0_{k+1}, \ldots, \vec{y}^0_M)$ at sites θ (and thus lower bounds $(\vec{lr}_{k+1}, \ldots, \vec{lr}_M)$), the following are equivalent:*

   (a) *There exists a compact split of Ω satisfying resource partition $\vec{B}_\theta$ and local consistency w.r.t. $(\vec{y}^0_{k+1}, \ldots, \vec{y}^0_M)$.*

   (b) *There exists a θ-permissible resource distribution w.r.t. $\vec{B}_\theta$ (with the above lower bounds).*

   (c) $\vec{lr}_\theta = \sum_{i \in \theta} \vec{lr}_i \leq \vec{B}_\theta$

2. *The combination of θ-permissible resource distribution w.r.t. $\vec{B}_\theta$ and $\bar{\theta}$-permissible*

resource distribution w.r.t. $\vec{B}_{\bar{\theta}}$ constitutes a permissible resource distribution.

Proof. (1) To prove this part, we first prove that (c) implies (a), (a) implies (b), and (b) implies (c).

(c) implies (a). Using the definition of $\vec{lr}_\theta$,

$$\begin{aligned}\vec{lr}_\theta \leq \vec{B}_\theta &\Leftrightarrow \sum_{i\in\theta} \vec{lr}_i \leq \vec{B}_\theta \\ &\Leftrightarrow \sum_{i\in\theta} A_i\vec{y}_i^0 \leq \vec{B}_\theta\end{aligned}$$

Then, selecting $\vec{r}_i = A_i\vec{y}_i^0$, for all i, $k+1 \leq i \leq M-1$, and $\vec{r}_M = \vec{B}_\theta - \sum_{i=k+1}^{M-1} \vec{r}_i$, we build a θ-compact split of Ω satisfying resource partition $\vec{B}_\theta$ and local consistency w.r.t. $(\vec{y}_{k+1}^0, \ldots, \vec{y}_M^0)$. Now, selecting $\vec{r}_i' = (\vec{b} - \vec{B}_\theta)/k$, for $1 \leq i \leq k$, $D(\vec{r}_1', \ldots, \vec{r}_k', \vec{r}_{k+1}, \ldots, \vec{r}_M)$ is a compact split of Ω. This completes this part of the proof.

(a) implies (b). Let $D(\vec{r}_1', \ldots, \vec{r}_M')$ be a compact split of Ω satisfying resource partition $\vec{B}_\theta$ and local consistency w.r.t. $(\vec{y}_{k+1}^0, \ldots, \vec{y}_M^0)$. Then, $\vec{lr}_i \leq \vec{r}_i'$ for all i, $k+1 \leq i \leq M$, and $\sum_{i\in\theta} \vec{r}_i' \leq \vec{B}_\theta$. Then, selecting $\vec{ur}_i = \vec{r}_i'$, for all i, $k+1 \leq i \leq M-1$, and $\vec{ur}_M = \vec{B}_\theta - \sum_{i=k+1}^{M-1} \vec{r}_i'$. Therefore, $\vec{lr}_i \leq \vec{r}_i' \leq \vec{ur}_i$, $k+1 \leq i \leq M$, i.e., $((\vec{lr}_{k+1}, \vec{r}_{k+1}', \vec{ur}_{k+1}), \ldots, (\vec{lr}_M, \vec{r}_M', \vec{ur}_M))$ is a permissible θ-resource distribution. This completes this part of the proof.

(b) implies (c). If there exists a θ-permissible resource distribution w.r.t. $\vec{B}_\theta$ (with

lower bounds $\vec{lr}_i = A\vec{y}_i^0$, for all i, $k+1 \leq i \leq M$). Then,

$$\vec{lr}_i \leq \vec{r}_i \leq \vec{ur}_i,\ k+1 \leq i \leq M \Rightarrow \sum_{i\in\theta} \vec{lr}_i \leq \sum_{i\in\theta} \vec{r}_i \leq \sum_{i\in\theta} \vec{ur}_i$$

$$\Leftrightarrow \sum_{i\in\theta} \vec{lr}_\theta \leq \vec{B}_\theta$$

This completes this part of the proof.

(2) Let $((\vec{lr}_1, \vec{r}_1, \vec{ur}_1), \ldots, (\vec{lr}_k, \vec{r}_k, \vec{ur}_k))$ and $((\vec{lr}_{k+1}, \vec{r}_{k+1}, \vec{ur}_{k+1}), \ldots, (\vec{lr}_M, \vec{r}_M, \vec{ur}_M))$ be a $\bar{\theta}$- and θ- permissible resource distribution w.r.t. $\vec{B}_{\bar{\theta}}$ and $\vec{B}_\theta$ respectively. Since, $\vec{B}_{\bar{\theta}} + \vec{B}_\theta = \vec{b}$, ($\bar{\theta}$-,$\theta$-) permissible resource distributions constitute a permissible resource distribution. □

The optimization functions f are defined in terms of the information of all sites (i.e., $D(\vec{r}_1, \ldots, \vec{r}_M)$), whereas we need to work with the information only on a subset θ of sites. To do that, we define the notion of θ-localizer as follows.

Definition 17. *Let Ω be a constraint over $\vec{x}$, and $\mathbb{S}$ be the set of all splits of Ω, $f : \mathbb{S} \to \mathbb{R}$ be a function. Let $\theta = \{k+1, \ldots, M\}$ be a subset of sites $\{1, \ldots, M\}$, $\bar{\theta}$ its complement, $\mathbb{S}_\theta$ be the set of all θ-splits. Then, function $f_\theta : \mathbb{S}_\theta \to \mathbb{R}$ is called θ-localizer of f if for any $\vec{r}_1^0, \ldots, \vec{r}_k^0$, and for every two splits $D(\vec{r}_1^0, \ldots, \vec{r}_k^0, \vec{r}_{k+1}, \ldots, \vec{r}_M)$ and $D(\vec{r}_1^0, \ldots, \vec{r}_k^0, \vec{r}'_{k+1}, \ldots, \vec{r}'_M)$ in $\mathbb{S}$,*

$$f(\vec{r}_1^0, \ldots, \vec{r}_k^0, \vec{r}_{k+1}, \ldots, \vec{r}_M) \geq f(\vec{r}_1^0, \ldots, \vec{r}_k^0, \vec{r}'_{k+1}, \ldots, \vec{r}'_M) \Leftrightarrow$$

$$f_\theta(\vec{r}_{k+1}, \ldots, \vec{r}_M) \geq f_\theta(\vec{r}'_{k+1}, \ldots, \vec{r}'_M)$$

(i.e., f_θ preserves monotonicity for any resource instantiation outside of θ).

We are now ready to formulate a theorem to be used for concurrent (re-) decompositions of the global constraints.

Theorem 5. *Let $\Omega = A\vec{x} \leq \vec{b}$ be a global constraints, θ be a subset, say $\{k+1, \ldots, M\}$, of sites $\{1, \ldots, M\}$ and $\bar{\theta}$ be its complement, $\mathbb{S}$ be the set of all compact splits of Ω, $f : \mathbb{S} \to \mathbb{R}$ be a monotonic function and f_θ be its θ-localizer, $\vec{B}_\theta$ be a resource bound partition, and $D(\vec{r}_1^0, \ldots, \vec{r}_k^0)$ be a (partial) $\bar{\theta}$-split for sites outside θ. Let Pr be a subset of properties that contains* rp *and $\Phi_{\theta-Pr}$ be the condition (1) $\vec{r}_\theta \leq \vec{B}_\theta$ for the case that Pr contains just* rp*, and the conditions (2) $\vec{r}_\theta \leq \vec{B}_\theta, \vec{lr}_i \leq \vec{r}_i$ for every $i \in \theta$, for the case of Pr that contains both* rp *and* lc*. Then,*

1. *For Pr being the set of properties $\{rp, pcp\}$ or $\{rp, pcp, lc\}$, let $(\vec{r}'_{k+1}, \ldots, \vec{r}'_M)$ be a solution to the problem*

$$max f_\theta(\vec{r}_{k+1}, \ldots, \vec{r}_M)$$

$$s.t. \Phi_{\theta-Pr}$$

Then, $(\vec{r}_1^0, \ldots, \vec{r}_k^0, \vec{r}'_{k+1}, \ldots, \vec{r}'_M)$ is a solution to the problem

$$max f(\vec{r}_1, \ldots, \vec{r}_M)$$

$$s.t. \Phi_{Pr}$$

2. *For Pr being the set of properties $\{rp\}$ or $\{rp, lc\}$, let $(\vec{r}'_{k+1}, \ldots, \vec{r}'_M)$ be a solution*

to the problem

$$max f_{\theta}(\vec{r}_{k+1}, \ldots, \vec{r}_M)$$

$$s.t. \Phi_{\theta-Pr}$$

and $(\vec{r}'_1, \ldots, \vec{r}'_k)$ *be a solution to the problem*

$$max f_{\bar{\theta}}(\vec{r}_{k+1}, \ldots, \vec{r}_M)$$

$$s.t. \Phi_{\bar{\theta}-Pr}$$

Then, $(\vec{r}'_1, \ldots, \vec{r}'_k, \vec{r}'_{k+1}, \ldots, \vec{r}'_M)$ *is a solution to the problem*

$$max f(\vec{r}_1, \ldots, \vec{r}_M)$$

$$s.t. \Phi_{Pr}$$

Proof. The proof follows from the fact that f has a θ-localizer, and from Propositions 8 and 9. □

The next subsection presents the analytical expression for the objective function, under uniformity assumptions on updates.