CSI 771 project report

----Replications and extensions of optimal computing budget allocation method by Chen (2000b)

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1. Introduction

Ordinal Optimization has emerged as an efficient technique for simulation and optimization. Exponential convergence rates can be achieved in many cases. In this paper, Chen et al. present a new approach, named Optimal Computing Budget Allocation (OCBA), which can further enhance the efficiency of ordinal optimization.

1.1. Optimal Optimization (OO)

Discrete-event system (DES) simulation is becoming increasingly important in the analysis and design of complicated man-made systems such as manufacturing plants, telecommunication networks, and traffic systems. Traditional optimization approaches focus on iteratively searching the design universe and converging to the best one. However, these approaches can be both expensive and time consuming due to their massive search space and their evolution in time according to complex man-made rules and influence of random occurrences. Even the simulation of a single design can be expensive because accurately estimating performance measures usually requires simulation until steady state in DES. Thus, finding the best design is often infeasible when the computing budget is very limited. Instead of insisting on picking the best design, Ordinal Optimization (Ho et al 1992) concentrates on finding good, or better, designs and reduces the required simulation time dramatically. More specifically, simulation stops when the obtained results are good enough or the confidence level is high enough. This idea has been applied on a 10-node network (Nikos et al 1994), where it is shown that we can isolate a good design with high probability with relatively short simulations, demonstrating many orders of magnitude of speedup. Furthermore, Chen (1996) presents a systematic and simple way to quantify confidence level for large DES simulation (the "large" refers to large search space), which is central to Ordinal Optimization and provides the basis to determine how to allocate computing budget among designs.

1.2. Optimal Computing Budget Allocation (OCBA)

Since the estimation of confidence level for large DES simulation is feasible, we may consider the DES optimization problem from another point of view: How could we optimally choose simulation lengths for different designs to minimize the total computation time while satisfying the desired confidence level?

Previous researchers have examined various approaches. Chen (1995) formulates the procedure of allocating computational efforts as a nonlinear optimization problem. Chen et al. (1996) apply the steepest-ascent method to solve the budget allocation problem. The major drawback of the steepest-ascent method is that an extra computation cost is needed to iteratively search for a solution to the budget allocation problem. Such an extra cost could be significant if the number of iterations is large. Chen et al. (1997) introduce a greedy heuristic to solve the budget allocation problem. This greedy heuristic iteratively determines which design appears to be the most promising for further simulation. However, the budget allocation selected by the greedy heuristic is not necessarily optimal.

On the other hand, instead of seeking a way to minimize the total computation time while ensuring a high probability of correctly selecting a good design, some new approaches intend to maximize the probability of correct selection under a fixed computing budget. Chen et al. (2000a) replace the objective function with an approximation and the use of Chernoff's bounds, and present an analytical solution to the approximation. The approach in Chen et al (2000a) provides a more efficient allocation than the greedy approach and the steepest-ascent method. Furthermore, Chen et al. (2000b) develop an asymptotically optimal approach (OCBA) for solving the budget allocation problem. It is even more efficient than the one given by Chen et al. (2000a). This is accomplished by replacing the objective function with a better approximation that can be solved analytically.

In this paper, we will extend the results in OCBA (Chen et al. 2000b) in the case that we have multiply best designs. The proposed approach is simple, general, practical and complementary to other techniques.

The paper is organized as follows: In section 2.1, we formulate the optimal computing budget allocation problem. Since our approach is based on the Bayesian model, we also provide a brief discussion of that model for completeness. Section 2.2 presents an asymptotic allocation rule for OCBA. Section 2.3 discusses an extension in the case that we have multiple best designs. The performance of the technique is illustrated with a series of numerical examples in Section 3. Section 4 concludes the paper.

2. Optimal Computing Budget Allocation (OCBA) Method and Extension

2.1. Problem Formulation

Suppose that our goal is to select a design associated with the smallest mean performance measure among k designs with unknown variances that are not necessarily equal. Further assume that the computing budget is limited and the number of designs is large. Denote

k: total number of designs;

Xij: jth i.i.d. sample of the performance measure from design i;

Ni: number of simulation replications for design i;

Xi: vector representing the simulation output for design i; $X_i = \{X_{ij} : j = 1, 2, \cdots, N_i\}$;

$\overline{\mu_i}$: sample mean performance measure for design i; $\overline{\mu_i} = \frac{1}{N_i}\sum_{j=1}^{N_i} X_{ij}$;

μ_i: mean performance measure; μ_i = E(Xij);

σ_i^2: variance for design i;

b: design having the smallest sample mean performance measure, i.e., $\overline{\mu_b} \le \min_i \overline{\mu_i}$;

$\delta_{b,i} \equiv \overline{\mu_b} - \overline{\mu_i}$

In ranking and selection problems, while the design with the smallest sample mean (design b) is usually picked, design b is not necessarily the one with the smallest unknown mean performance. Correct selection is therefore defined as the event that design b is actually the best design (i.e., with the smallest population performance). In the remainder of this paper, let "CS" denote Correct Selection. Thus,

$$P\{CS\} = P\{design\ b\ is\ actually\ the\ best\ design\} = P\{\mu_b < \mu_i, i \ne b\} \qquad (1)$$

We wish to minimize the total computation cost while obtaining a desired confidence level in selecting the best system out of k competing designs. If simulations are performed on a sequential computer, the computation cost can be approximated by ($N_1 + N_2 + \ldots + N_k$). Ideally we want

$$\min_{N_1, N_2, \cdots, N_k} (N_1 + N_2 + \cdots + N_k) \qquad (2)$$
$$s.t. \quad P\{CS\} \ge P^*$$

where P^* is a user-defined confidence level requirement.

To solve problem (2), we must be able to estimate P\{CS\}. There exists a large literature on assessing P\{CS\} based on classical statistical models (e.g., Goldsman and Nelson 1994, Banks 1998 give an excellent survey on available approaches). However, most of these approaches are only suitable for problems with a small number of designs. Recently, Chen (1996) introduced an estimation technique that approximates P\{CS\} for ordinal comparison when the number of designs is large based on a Bayesian model (Bernardo and Smith 1994). This technique has the added benefit of providing sensitivity information that is useful in solving problem (2). We will incorporate this technique within our budget allocation approach.

Many performance measures of interest are taken over some averages of a sample path or a batch of samples. Thus, the simulation output tends to be normally distributed in many applications. In this paper we assume that the simulation output, Xij, is normally distributed. After the simulation is performed, a posterior distribution of μ_i, $p(\mu_i \mid X_i)$ can be constructed based on two pieces of information: (i) prior knowledge of the system's performance, and (ii) current simulation output. If we select the observed best design (design b), the probability that we selected the best design is

$$P\{CS\} = P(\mu_b < \mu_i, i \ne b \mid X_i, i = 1, 2, \cdots, k) .$$

To simplify the notation used, we rewrite Eq. (1) as $P\{\widehat{\mu_b} < \widehat{\mu_i},\ i \ne b\}$, where $\widehat{\mu_i}$ denotes the random variable whose probability distribution is the posterior distribution for design i. Assume that the unknown mean μ_i has the conjugate normal prior distribution. We consider non-informative prior distributions. This implies that no priori knowledge is given about the performance of any design alternative before conducting the simulation. In that case, DeGroot (1970) shows that the posterior distribution of μ_i is

$$\widehat{\mu_i} \sim N(\overline{\mu_i}, \frac{\sigma_i^2}{N_i}).$$

After the simulation is performed, $\overline{\mu_i}$ can be calculated, σ_i^2 can be approximated by the sample variance; $P\{\text{CS}\}$ can then be estimated using a Monte Carlo simulation. However, estimating $P\{\text{CS}\}$ via Monte Carlo simulation is time-consuming. Since the purpose of budget allocation is to improve simulation efficiency, we need a relatively fast and inexpensive way of estimating $P\{\text{CS}\}$ within the budget allocation procedure. Efficiency is more crucial than estimation accuracy in this setting. We adopt a common approximation procedure used in simulation and statistics literature (Brately et al. 1987, Chick 1997, Law and Kelton 1991). This approximation is based on the Bonferroni inequality.

Let Y_i be a random variable. According to the Bonferroni inequality, $P\{\bigcap_{i=1}^{k}(Y_i < 0)\} \geq 1 - \sum_{i=1}^{k}[1 - P\{Y_i < 0\}]$. In our case, Y_i is replaced by $(\widehat{\mu_b} - \widehat{\mu_i})$ to provide a lower bound for the probability of correct selection. That is,

$$P\{CS\} = P\{\bigcap_{i=1, i\neq b}^{k}(\widehat{\mu_b} - \widehat{\mu_i} < 0)\} \geq 1 - \sum_{i=1, i\neq b}^{k}[1 - P\{\widehat{\mu_b} - \widehat{\mu_i} < 0\}] = 1 - \sum_{i=1, i\neq b}^{k} P\{\widehat{\mu_b} > \widehat{\mu_i}\} = APCS$$

We refer to this lower bound of the correct selection probability as the *Approximate Probability of Correct Selection* (*APCS*). *APCS* can be computed very easily and quickly; no extra Monte Carlo simulation is needed. Numerical tests show that the APCS approximation can still lead to highly efficient procedures (e.g., Chen 1996, Inoue and Chick 1998). We therefore use *APCS* to approximate $P\{\text{CS}\}$ as the simulation experiment proceeds. More specifically, we consider the following problem:

$$\begin{aligned} &\min_{N_1, N_2, \cdots, N_k} (N_1 + N_2 + \cdots + N_k) \\ &s.t. \quad 1 - \sum_{i=1, i\neq b}^{k} P\{\widehat{\mu_b} > \widehat{\mu_i}\} \geq P^* \end{aligned} \tag{3}$$

In the next section, an asymptotic allocation rule with respect to the number of simulation replications, N_i will be presented.

2.2. An Asymptotic Allocation Rule

Obviously, for Eq. (3), large P^* requires large $(N_1 + N_2 + \cdots + N_k)$, hence, we consider the following:

$$\begin{aligned} &\min_{N_1, N_2, \cdots, N_k} (N_1 + N_2 + \cdots + N_k) \\ &s.t. \quad 1 - \sum_{i=1, i\neq b}^{k} P\{\widehat{\mu_b} > \widehat{\mu_i}\} = P^* \end{aligned} \tag{4}$$

For the s.t.,

$$\sum_{i=1, i\neq b}^{k} P\{\widehat{\mu_b} > \widehat{\mu_i}\} = \sum_{\substack{i=1 \\ i\neq b}}^{k} \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_{b,i}} e^{-\frac{(x-\delta_{b,i})^2}{2\sigma_{b,i}^2}} dx = \sum_{\substack{i=1 \\ i\neq b}}^{k} \int_{-\frac{\delta_{b,i}}{\sigma_{b,i}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt,$$

where $\delta_{b,i} = \widehat{\mu_b} - \widehat{\mu_i}$, and $\sigma_{b,i}^2 = \frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}$, for notation simplification. Then, let F be the Lagrangian relaxation of (4):

$$F = \sum_{i=1}^{k} N_i + \lambda[1 - \sum_{\substack{i=1 \\ i \neq b}}^{k} \int_{-\frac{\delta_{b,i}}{\sigma_{b,i}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt - P^*].$$

Furthermore, the conditions of this problem can be stated as follows:

$$\frac{\partial F}{\partial N_i} = 1 - \lambda \frac{\partial \int_{-\frac{\delta_{b,i}}{\sigma_{b,i}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}{\partial(-\frac{\delta_{b,i}}{\sigma_{b,i}})} \frac{\partial(-\frac{\delta_{b,i}}{\sigma_{b,i}})}{\partial \sigma_{b,i}} \frac{\partial \sigma_{b,i}}{\partial N_i}$$

$$= 1 - \lambda(-\frac{1}{\sqrt{2\pi}} e^{-\frac{(-\frac{\delta_{b,i}}{\sigma_{b,i}})^2}{2}})(\frac{\delta_{b,i}}{\sigma_{b,i}^2})(-\frac{1}{2\sigma_{b,i}} \frac{\sigma_i^2}{N_i^2}) \tag{5}$$

$$= 1 - \frac{\lambda}{2\sqrt{2\pi}} e^{-\frac{\delta_{b,i}^2}{2\sigma_{b,i}^2}} \frac{\delta_{b,i}\sigma_i^2}{(\sigma_{b,i}^2)^{\frac{3}{2}} N_i^2} = 0, \ for \ i = 1, 2, \cdots, k, \ and \ i \neq b.$$

$$\frac{\partial F}{\partial N_b} = 1 - \frac{\lambda}{2\sqrt{2\pi}} \sum_{\substack{i=1 \\ i \neq b}}^{k} e^{-\frac{\delta_{b,i}^2}{2\sigma_{b,i}^2}} \frac{\delta_{b,i}\sigma_b^2}{(\sigma_{b,i}^2)^{\frac{3}{2}} N_b^2} = 0, \tag{6}$$

We now examine the relationship between N_b and N_i for i = 1, 2, ..., k, and $i \neq b$. From Eq.(5) and (6), we have

$$\sum_{\substack{i=1 \\ i \neq b}}^{k} \frac{N_i^2}{\sigma_i^2} = \frac{N_b^2}{\sigma_b^2} \tag{7}$$

Then

$$N_b = \sigma_b \sqrt{\sum_{\substack{i=1 \\ i \neq b}}^{k} \frac{N_i^2}{\sigma_i^2}} \tag{8}$$

We further investigate the relationship between *Ni* and *Nj*, for any $i, j \in \{1, 2, ..., k\}$, and $i \neq j \neq b$. From Eq. (5),

$$e^{-\frac{\delta_{b,i}^2}{2(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i})}} \frac{\delta_{b,i}\sigma_i^2}{(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i})^{\frac{3}{2}} N_i^2} = e^{-\frac{\delta_{b,j}^2}{2(\frac{\sigma_b^2}{N_b} + \frac{\sigma_j^2}{N_j})}} \frac{\delta_{b,j}\sigma_j^2}{(\frac{\sigma_b^2}{N_b} + \frac{\sigma_j^2}{N_j})^{\frac{3}{2}} N_j^2} \tag{9}$$

To reduce the total simulation time for identifying a good design, it is worthwhile to concentrate the computational effort on good designs. Namely, *Nb* should be increased relative to *Ni*, for *i* = 1, 2, ..., *k*, and $i \neq b$. This is indeed the case in the actual simulation experiments. And this assumption can be supported by considering a special case in Eq. (5): When $\sigma 1 = \sigma 2 = ... = \sigma k$,

$$N_b = \sqrt{\sum_{\substack{i=1\\ i\neq b}}^{k} N_i^2}$$

Therefore, we assume that *Nb* >>*Ni*, which enables us to simplify Eq. (9) as

$$e^{-\frac{\delta_{b,i}^2}{2(\frac{\sigma_i^2}{N_i})}} \frac{\delta_{b,i}\sigma_i^2}{(\frac{\sigma_i^2}{N_i})^{\frac{3}{2}} N_i^2} = e^{-\frac{\delta_{b,j}^2}{2(\frac{\sigma_j^2}{N_j})}} \frac{\delta_{b,j}\sigma_j^2}{(\frac{\sigma_j^2}{N_j})^{\frac{3}{2}} N_j^2}$$

On rearranging terms, the above equation becomes

$$e^{\frac{1}{2}(\frac{\delta_{b,j}^2}{\frac{\sigma_j^2}{N_j}} - \frac{\delta_{b,i}^2}{\frac{\sigma_i^2}{N_i}})} \frac{N_j^{\frac{1}{2}}}{N_i^{\frac{1}{2}}} = \frac{\delta_{b,j}\sigma_i}{\delta_{b,i}\sigma_j}$$

Taking the natural log on both sides, we have

$$\frac{\delta_{b,j}^2}{\sigma_j^2} N_j + \ln(N_j) = \frac{\delta_{b,i}^2}{\sigma_i^2} N_i + \ln(N_i) + 2\ln(\frac{\delta_{b,j}\sigma_i}{\delta_{b,i}\sigma_j})$$

Usually, the user-defined confidence level requirement P^* is high and hence $N_1 + N_2 + \cdots + N_k$ is large. To further facilitate the computations, we intend to find an asymptotic allocation rule. Namely, we consider the case that $N_1 + N_2 + \cdots + N_k \to \infty$. In this case, all the ln terms become much smaller than the other terms and are negligible. This implies

$$\frac{\delta_{b,j}^2}{\sigma_j^2} N_j = \frac{\delta_{b,i}^2}{\sigma_i^2} N_i$$

Therefore, we obtain the ratio between Ni and Nj as:

$$\frac{N_i}{N_j} = (\frac{\sigma_i / \delta_{b,i}}{\sigma_j / \delta_{b,j}})^2, \text{ for } i = 1, 2, \ldots, k, \text{ and } i \neq j \neq b. \qquad (10)$$

We therefore have the following result:

Theorem 1. For the problem to choose N_1, N_2, ··, N_k simulation samples to be allocated to k competing designs whose performance is depicted by random variables with means $\mu_1, \mu_2, \cdots, \mu_k$, and finite variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$ respectively, subject to the restriction that P{CS} is greater than some satisfactory level P*, as P* $\to 1$ or $N_1 + N_2 + \cdots + N_k \to \infty$, the total computation cost is asymptotically minimized when

(1) $\frac{N_i}{N_j} = (\frac{\sigma_i / \delta_{b,i}}{\sigma_j / \delta_{b,j}})^2$, for $i = 1, 2, \ldots, k$, and $i \neq j \neq b$

(2) $N_b = \sqrt{\sum_{\substack{i=1\\ i\neq b}}^{k} N_i^2}$

where N_i is the number of samples allocated to design *i*, $\delta_{b,i} \equiv \overline{\mu_b} - \overline{\mu_i}$, and $\overline{\mu_b} \leq \min_i \overline{\mu_i}$.

With Theorem 1, we now present a cost-effective sequential approach based on OCBA to select the best design from *k* alternatives subject to that P(CS) must be no less than P*. Initially, n_0 simulation replications for each of *k* design are conducted to get some information about the performance of each design during the first stage. As simulation proceeds, the sample means and sample variances of each design are computed from the data already collected up to that stage. According to this collected simulation output, an incremental computing budget, Δ, is allocated based on Theorem 1 at each stage. Ideally, each new replication should bring us closer to the optimal solution. This procedure is continued until P(CS) is no less than P*. The algorithm is summarized as follows.

A Sequential Algorithm for Optimal Computing Budget Allocation (OCBA)

Step 0. Perform *n*0 simulation replications for all designs; *l*←0; $N_1^l = N_2^l = \cdots = N_k^l = n_0$.

Step 1. For *i* = 1, 2, ..., *k*, calculate $APCS(N_1^l, N_2^l, \cdots, N_k^l)$. If $APCS(N_1^l, N_2^l, \cdots, N_k^l) \geq P^*$, stop.

Step 2. Increase the computing budget (i.e., number of additional simulations) by Δ and compute the new budget allocation, $N_1^{l+1}, N_2^{l+1}, \cdots, N_k^{l+1}$, using (8) and (10).

Step 3. Perform additional max $(0, N_i^{l+1} - N_i^l)$ simulations for design *i*, *i* = 1, ..., *k*. *l*← *l* + 1. Go to Step 1.

In the above algorithm, l is the iteration number. As simulation evolves, design b, which is the design with the largest sample mean, may change from iteration to iteration, although it will converge to the optimal design as the l goes to infinity. When b changes, Theorem 1 is directly applied in step 2. However, the older design b may not be simulated at all in this iteration in step 3 due to extra allocation to this design in earlier iterations.

In addition, we need to select the initial number of simulations, n0, and the one-time increment, Δ. Chen et al. (1999) offers detailed discussions on the selection. It is well understood that n0 cannot be too small as the estimates of the mean and the variance may be very poor, resulting in premature termination of the comparison. A suitable choice for n0 is between 5 and 20 (Law and Kelton 1991, Bechhofer et al. 1995). Also, a large Δ can result in waste of computation time to obtain an unnecessarily high confidence level. On the other hand, if Δ is small, we need to the computation procedure in step 2 many times. A suggested choice for Δ is a number bigger than 5 but smaller than 10% of the simulated designs.

2.3. *Extension for M best designs (OCBAM method)*

If we have multiple best designs, in Eq. (10), $\delta_{b,i}$ converges to 0 for some i, for $i = 1, 2, \ldots, k$, and $i \neq b$ as $N_1 + N_2 + \cdots + N_k \to \infty$. And the allocation procedure in Eq. (10) will allocate a great amount of simulation budget to design i and design b in order to separate them, which is impossible because they have same performance. Obviously, this is not a good strategy because what we often need is to choose all of the best designs or one of them if we have multiple best designs. The Optimal Computing Budget Allocation procedure to choose all the M best designs (OCBAM) is as follows.

Problem statement: Among k designs, we assume that we already know that there are m best designs, which have same performance. We want to find these m best designs.

Designs are sequenced according to their sample mean. That is, $\overline{J_1} \le \overline{J_2} \le \cdots\cdots \le \overline{J_k}$. Also, we assume that the posterior distribution of design i: $\widetilde{J_i} \sim N(\overline{J_i}, \frac{\sigma_i^2}{N_i})$

P (CS) definition:

$$P\{CS\} = P\{\bigcap_{1\le i\le m, m+1\le j\le k}^{k} (\widetilde{J_i} - \widetilde{J_j} < 0)\}$$

$$\ge 1 - \sum_{i=1}^{m}\sum_{j=m+1}^{k} [1 - P\{\widetilde{J_i} - \widetilde{J_j} < 0\}] \tag{11}$$

$$= 1 - \sum_{i=1}^{m}\sum_{j=m+1}^{k} P\{\widetilde{J_i} > \widetilde{J_j}\} = APCS$$

Objective function:

$$\max_{N_1, N_2, \cdots, N_k} 1 - \sum_{i=1}^{m}\sum_{j=m+1}^{k} P\{\widetilde{J_i} > \widetilde{J_j}\}$$

$$s.t. \sum_{i=1}^{k} N_i = T \ \ and \ \ N_i \ge 0. \tag{12}$$

For the objective function,

$$\sum_{i=1}^{m}\sum_{j=m+1}^{k} P\{\widetilde{J_i} > \widetilde{J_j}\}$$

$$= \sum_{i=1}^{m}\sum_{j=m+1}^{k} \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma_{i,j}} e^{-\frac{(x-\delta_{i,j})^2}{2\sigma_{i,j}^2}} dx \tag{13}$$

$$= \sum_{i=1}^{m}\sum_{j=m+1}^{k} \int_{-\frac{\delta_{i,j}}{\sigma_{i,j}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

Where $\delta_{i,j} = \widetilde{J_i} - \widetilde{J_j}$ and $\sigma_{i,j}^2 = \frac{\sigma_i^2}{N_i} + \frac{\sigma_j^2}{N_j}$, for notation simplification. Then, let F be the Lagrangian relaxation of (12):

$$F = 1 - \sum_{i=1}^{m}\sum_{j=m+1}^{k} \int_{-\frac{\delta_{i,j}}{\sigma_{i,j}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt - \lambda(\sum_{i=1}^{k} N_i - T). \tag{14}$$

Furthermore,

$$\frac{\partial F}{\partial N_j} = -\sum_{i=1}^{m} \frac{\partial \int_{-\frac{\delta_{i,j}}{\sigma_{i,j}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}{\partial(-\frac{\delta_{i,j}}{\sigma_{i,j}})} \frac{\partial(-\frac{\delta_{i,j}}{\sigma_{i,j}})}{\partial \sigma_{i,j}} \frac{\partial \sigma_{i,j}}{\partial N_j} - \lambda$$

$$= -\sum_{i=1}^{m} (-\frac{1}{\sqrt{2\pi}} e^{-\frac{(-\frac{\delta_{i,j}}{\sigma_{i,j}})^2}{2}})(\frac{\delta_{i,j}}{\sigma_{i,j}^2})(-\frac{1}{2\sigma_{i,j}} \frac{\sigma_j^2}{N_j^2}) - \lambda \tag{15}$$

$$= -\frac{1}{2\sqrt{2\pi}} \sum_{i=1}^{m} e^{-\frac{\delta_{i,j}^2}{2\sigma_{i,j}^2}} \frac{\delta_{i,j}\sigma_j^2}{(\sigma_{i,j}^2)^{\frac{3}{2}} N_j^2} - \lambda = 0, \ for \ \ j = m+1, m+2, \cdots, k.$$

$$\frac{\partial F}{\partial N_i} = -\frac{1}{2\sqrt{2\pi}} \sum_{j=m+1}^{k} e^{-\frac{\delta_{i,j}^2}{2\sigma_{i,j}^2}} \frac{\delta_{i,j}\sigma_i^2}{(\sigma_{i,j}^2)^{\frac{3}{2}} N_i^2} - \lambda = 0 \tag{16}$$

$$\lambda(\sum_{i=1}^{k} N_i - T) = 0, \ and \ \ \lambda \geq 0. \tag{17}$$

We now examine the relationship between N_i and N_j for i = 1, 2, ..., k, and $i \neq b$. From Eq(15), we have

$$\frac{1}{2\sqrt{2\pi}} \sum_{i=1}^{m} e^{-\frac{\delta_{i,j}^2}{2\sigma_{i,j}^2}} \frac{\delta_{i,j}}{(\sigma_{i,j}^2)^{\frac{3}{2}}} = -\lambda \frac{N_j^2}{\sigma_j^2} \tag{18}$$

Then, we have

$$\frac{1}{2\sqrt{2\pi}} \sum_{j=m+1}^{k} \sum_{i=1}^{m} e^{-\frac{\delta_{i,j}^2}{2\sigma_{i,j}^2}} \frac{\delta_{i,j}}{(\sigma_{i,j}^2)^{\frac{3}{2}}} = -\lambda \sum_{j=m+1}^{k} \frac{N_j^2}{\sigma_j^2} \tag{19}$$

From Eq(16), we have

$$\frac{1}{2\sqrt{2\pi}} \sum_{j=m+1}^{k} e^{-\frac{\delta_{i,j}^2}{2\sigma_{i,j}^2}} \frac{\delta_{i,j}}{(\sigma_{i,j}^2)^{\frac{3}{2}}} = -\lambda \frac{N_i^2}{\sigma_i^2} \tag{20}$$

Then, we have

$$\frac{1}{2\sqrt{2\pi}} \sum_{i=1}^{m} \sum_{j=m+1}^{k} e^{-\frac{\delta_{i,j}^2}{2\sigma_{i,j}^2}} \frac{\delta_{i,j}}{(\sigma_{i,j}^2)^{\frac{3}{2}}} = -\lambda \sum_{i=1}^{m} \frac{N_i^2}{\sigma_i^2} \tag{21}$$

From (19) and(21), we have

$$\sum_{i=1}^{m} \frac{N_i^2}{\sigma_i^2} = \sum_{j=m+1}^{k} \frac{N_j^2}{\sigma_j^2} \tag{22}$$

We only consider the case m<<k, so we can use the assumption in OCBA paper, that is Ni >>Nj, which allow $\frac{\sigma_j^2}{N_j}$ to approximate $\sigma_{i,j}^2 = \frac{\sigma_i^2}{N_i} + \frac{\sigma_j^2}{N_j}$. Further, since as $T \to \infty$, $\overline{J_i} = \overline{J_1}$, we have $\delta_{i,j} = \delta_{1,j}$. Therefore, we simplify Eq. (18) as

$$\frac{1}{2\sqrt{2\pi}} \times m \times e^{-\frac{\delta_{1,j}^2}{2\frac{\sigma_j^2}{N_j}}} \frac{\delta_{1,j}}{(\frac{\sigma_j^2}{N_j})^{\frac{3}{2}}} = -\lambda \frac{N_j^2}{\sigma_j^2} \quad (23)$$

By the same approach in the OCBA paper, we have

$$\frac{N_j}{N_l} = (\frac{\sigma_j / \delta_{1,j}}{\sigma_l / \delta_{1,l}})^2, \text{ for } m+1 \le j, l \le k \quad (24)$$

We further investigate the relationship between Ni and $Nh\,(1 \le i \le m, 1 \le h \le m)$. We substitute $\frac{\sigma_j^2}{N_j}$ for $\sigma_{i,j}^2$ in(20).

$$\frac{1}{2\sqrt{2\pi}} \sum_{j=m+1}^{k} e^{-\frac{\delta_{i,j}^2}{2\frac{\sigma_j^2}{N_j}}} \frac{\delta_{i,j}}{(\frac{\sigma_j^2}{N_j})^{\frac{3}{2}}} = -\lambda \frac{N_i^2}{\sigma_i^2} \quad (25)$$

Furthermore, $\delta_{i,j} = \delta_{1,j}$, which enable us to further simplify (25) as

$$\frac{1}{2\sqrt{2\pi}} \sum_{j=m+1}^{k} e^{-\frac{\delta_{1,j}^2}{2\frac{\sigma_j^2}{N_j}}} \frac{\delta_{1,j}}{(\frac{\sigma_j^2}{N_j})^{\frac{3}{2}}} = -\lambda \frac{N_i^2}{\sigma_i^2} \quad (26)$$

Therefore, from(26), we have

$$\frac{N_i^2}{\sigma_i^2} = \frac{N_h^2}{\sigma_h^2}, \text{ for } 1 \le i, h \le m \quad (27)$$

From(22), (24) and(27), we get all the allocation procedures we need.

Conclusion:

Given a total number of simulation samples T to be allocated to k competing designs whose performance is depicted by random variables with means $J(\theta_1), J(\theta_2), \cdots, J(\theta_k)$, and finite variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2$ respectively. There are m same best designs among them. As $T \to \infty$, the APCS can be asymptotically maximized when

1) $\frac{N_j}{N_l} = (\frac{\sigma_j / \delta_{1,j}}{\sigma_l / \delta_{1,l}})^2, \text{ for } m+1 \le j, l \le k$

2) $\frac{N_i^2}{\sigma_i^2} = \frac{1}{m} \sum_{j=m+1}^{k} \frac{N_j^2}{\sigma_j^2}$, for $1 \le i \le m$

And we call this procedure as Optimal Computing Budget Allocation procedure to choose all the M best designs (OCBAM).

3. Numerical Testing Replication and Extension

We have 4 experiments in this section. Our experiment 1 and experiment 2 correspond to original paper's experiment 1 and experiment 5. Instead of comparing OCBA procedure with 4 different other procedure, we only compared OCBA with Equal allocation procedure. Our experiment 3 and experiment 4 are extensions for the case that we have M same best designs. All designs are under normal distribution.

We estimate the P{CS} by counting the number of times in which we successfully find the true best design (design 0 in all of our examples) out of total independent applications of each selection procedure. P {CS} is then obtained by dividing this number by total number of replication, representing the correct selection frequency. The number of replication is 10000 for experiment 1 and experiment 2, and 1000 for experiment 3 and experiment 4.

We set $\Delta = 20$ and $n0 = 10$ for all 4 experiments.

<u>Experiment 1, Normal Distribution, 10 designs</u>

There are 10 design alternatives. Suppose Xi ~ N(i, 6^2), i = 0, 1, 2, .., 9. We want to find a design with the minimum mean. It is obvious that design 0 is the actual best design.

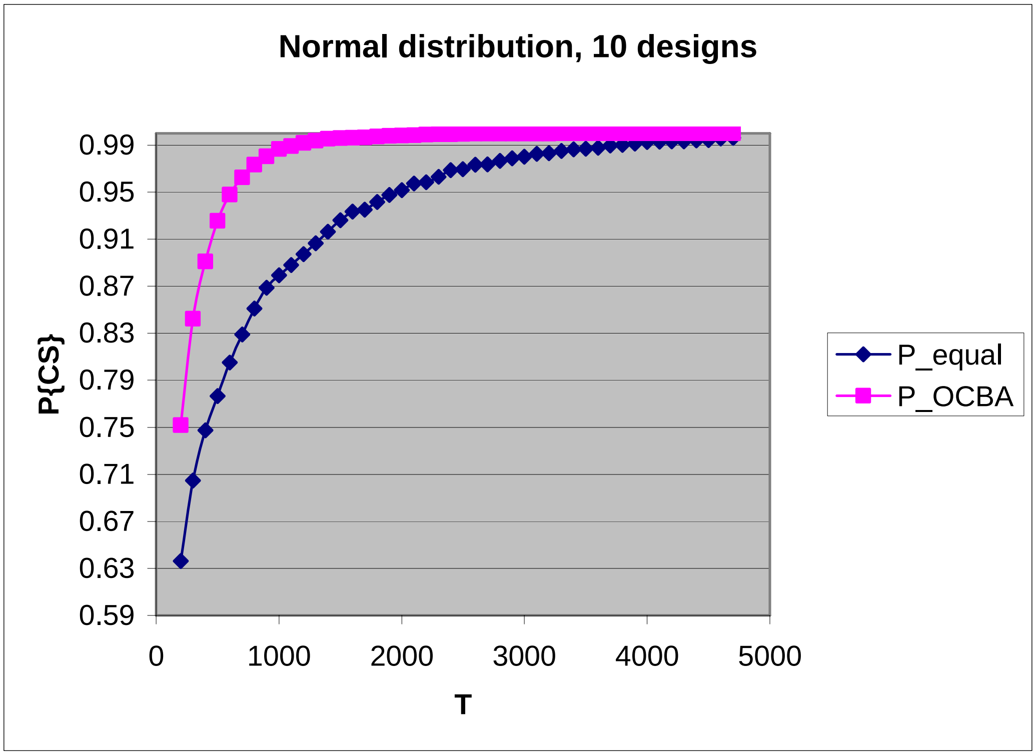
Different computing budgets are tested. Figure 1 shows the test results using OCBA and equal allocation procedure. Note that the simulation variance of each design is 36, while the difference of two adjacent designs' means is only 1. Given such a high noise ratio, we see that with the total computation cost as low as 1200 simulation samples, the probability of correctly selecting the best design is already higher than 99%. This demonstrates the advantage of applying OCBA procedure. On the other hand, equal allocation procedure needs 3800 computational cost to get the same P{CS}. Speedup factor is around 3800/1200 =3.17, which is similar to that of the paper (4400/1100 = 4).

Table 1. Normal distributions, 10 designs
T: simulation budget;
P_equal: P{CS} by equal allocation procedure;
P_OCBA: P{CS} by OCBA procedure.

T	P_equal	P_OCBA
200	0.6362	0.7517
300	0.7047	0.8423
400	0.7474	0.891
500	0.7766	0.9254
600	0.805	0.9477
700	0.829	0.9625
800	0.8511	0.9733

900	0.8687	0.9804
1000	0.8792	0.9868
1100	0.8881	0.9892
1200	0.8971	0.9918
1300	0.9064	0.9937
1400	0.9163	0.9954
1500	0.926	0.9959
1600	0.9333	0.9963
1700	0.9349	0.9966
1800	0.9416	0.9973
1900	0.9475	0.9978
2000	0.9516	0.998
2100	0.9572	0.9983
2200	0.9583	0.999
2300	0.9631	0.9992
2400	0.9687	0.9993
2500	0.9696	0.9994
2600	0.9733	0.9996
2700	0.9736	0.9996
2800	0.9765	0.9996
2900	0.9788	0.9996
3000	0.9801	0.9996
3100	0.9825	0.9997
3200	0.9832	0.9997
3300	0.9851	0.9999
3400	0.9863	1
3500	0.987	1
3600	0.9877	1
3700	0.9896	1
3800	0.9902	1
3900	0.9912	1
4000	0.9926	1
4100	0.9928	1
4200	0.9933	1
4300	0.9933	1
4400	0.9941	1
4500	0.9942	1
4600	0.9956	1
4700	0.9963	1

Figure 1. Normal distributions, 10 designs
P{CS} vs. T using OCBA and equal allocation procedure


Normal distribution, 10 designs
0.99
0.95
0.91
0.87
0.83
0.79
0.75
0.71
0.67
0.63
0.59
P{CS}
0
1000
2000
3000
4000
5000
T
P_equal
P_OCBA

Experiment 2, Normal Distribution, Bigger Design Space (100 designs)

There are 100 design alternatives. Suppose Xi ~ N (i/10, 1^2), i = 0, 1, 2, .., 9. We want to find a design with the minimum mean. It is obvious that design 0 is the actual best design.

Different computing budgets are tested. Figure 2 shows the test results using OCBA and equal allocation procedure. We see that with the total computation cost as 3000 simulation samples, the probability of correctly selecting the best design is 97%. On the other hand, equal allocation procedure uses 61000 computational cost to get P{CS}=96.5%. Speedup factor is at least 61000/3000 =20, which is much higher than that of experiment 1 (speedup factor is 3 ~ 4). Reason is given by this paper that a larger design space gives the OCBA algorithm more flexibility in allocating the computing budget.

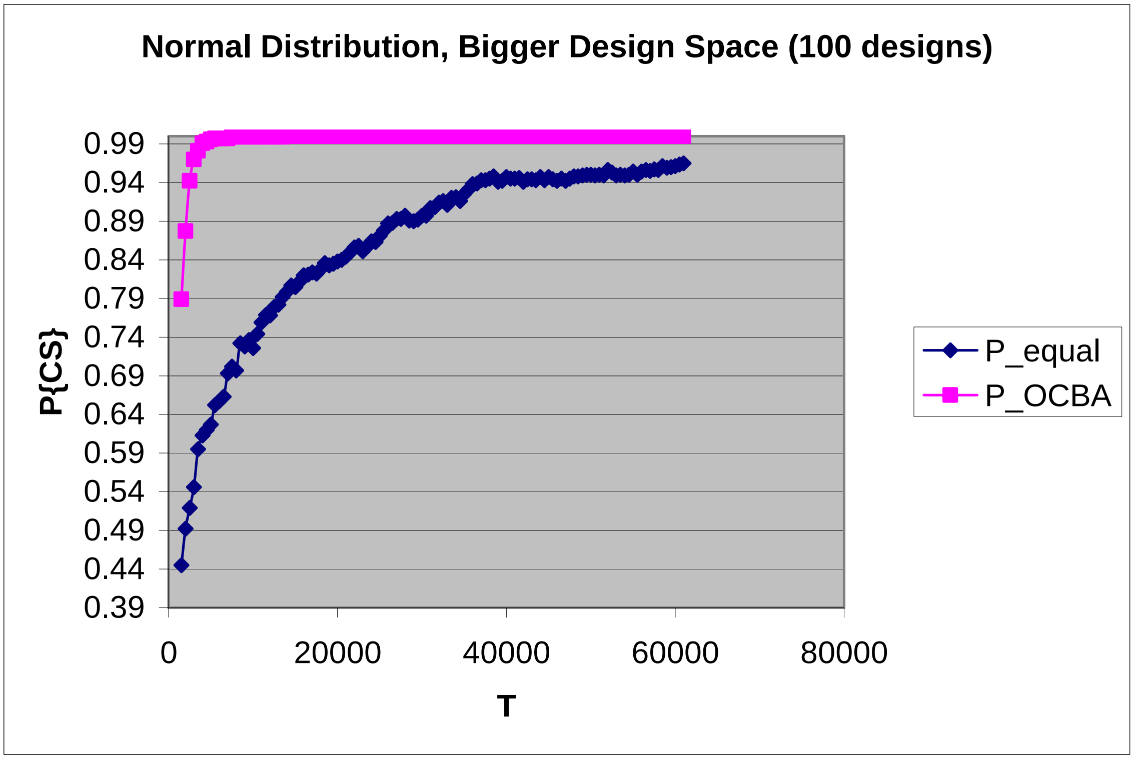
Table 2. Normal distributions, 100 designs
T: simulation budget;
P_equal: P{CS} by equal allocation procedure;
P_OCBA: P{CS} by OCBA procedure.

T	P_equal	P_OCBA
1500	0.445	0.789
2000	0.492	0.877
2500	0.519	0.942
3000	0.546	0.97
3500	0.595	0.981
4000	0.613	0.991
4500	0.62	0.993
5000	0.627	0.996
5500	0.652	0.997
6000	0.657	0.997
6500	0.663	0.997
7000	0.693	0.997
7500	0.702	0.999
8000	0.697	0.999
8500	0.732	0.999
9000	0.728	0.999
9500	0.736	0.999
10000	0.726	0.999
10500	0.744	0.999
11000	0.759	0.999
11500	0.769	0.999
12000	0.768	0.999
12500	0.779	0.999
13000	0.782	0.999
13500	0.792	0.999
14000	0.799	1
14500	0.807	1
15000	0.805	1
15500	0.812	1

16000	0.82	1
16500	0.821	1
17000	0.824	1
17500	0.822	1
18000	0.828	1
18500	0.836	1
19000	0.833	1
19500	0.835	1
20000	0.838	1
20500	0.84	1
21000	0.844	1
21500	0.85	1
22000	0.856	1
22500	0.858	1
23000	0.851	1
23500	0.857	1
24000	0.864	1
24500	0.863	1
25000	0.871	1
25500	0.878	1
26000	0.887	1
26500	0.888	1
27000	0.893	1
27500	0.893	1
28000	0.897	1
28500	0.891	1
29000	0.89	1
29500	0.892	1
30000	0.897	1
30500	0.897	1
31000	0.907	1
31500	0.908	1
32000	0.914	1
32500	0.916	1
33000	0.911	1
33500	0.92	1
34000	0.921	1
34500	0.916	1
35000	0.925	1
35500	0.931	1
36000	0.938	1
36500	0.939	1
37000	0.943	1
37500	0.943	1
38000	0.945	1
38500	0.948	1
39000	0.941	1
39500	0.942	1

40000	0.947	1
40500	0.945	1
41000	0.945	1
41500	0.946	1
42000	0.941	1
42500	0.944	1
43000	0.944	1
43500	0.943	1
44000	0.947	1
44500	0.943	1
45000	0.947	1
45500	0.944	1
46000	0.942	1
46500	0.945	1
47000	0.942	1
47500	0.945	1
48000	0.948	1
48500	0.948	1
49000	0.949	1
49500	0.95	1
50000	0.95	1
50500	0.949	1
51000	0.95	1
51500	0.949	1
52000	0.956	1
52500	0.953	1
53000	0.949	1
53500	0.95	1
54000	0.949	1
54500	0.95	1
55000	0.954	1
55500	0.95	1
56000	0.954	1
56500	0.956	1
57000	0.955	1
57500	0.957	1
58000	0.956	1
58500	0.961	1
59000	0.959	1
59500	0.96	1
60000	0.961	1
60500	0.963	1
61000	0.965	1

Figure 2. Normal distributions, 100 designs
P{CS} vs. T using OCBA and equal allocation procedure


Normal Distribution, Bigger Design Space (100 designs)
P{CS}
0.99
0.94
0.89
0.84
0.79
0.74
0.69
0.64
0.59
0.54
0.49
0.44
0.39
0
20000
40000
60000
80000
T
P_equal
P_OCBA

Experiment 3, Normal Distribution, 10 designs with 3 best designs

There are 10 design alternatives. Suppose Xi ~ N $(2, 6^2)$, i = 0, 1, 2 and Xi ~ N $(i, 6^2)$, i = 3, 4, .., 9. We want to find 3 designs with the minimum mean. It is obvious that design 0, 1, 2 are the actual best designs.

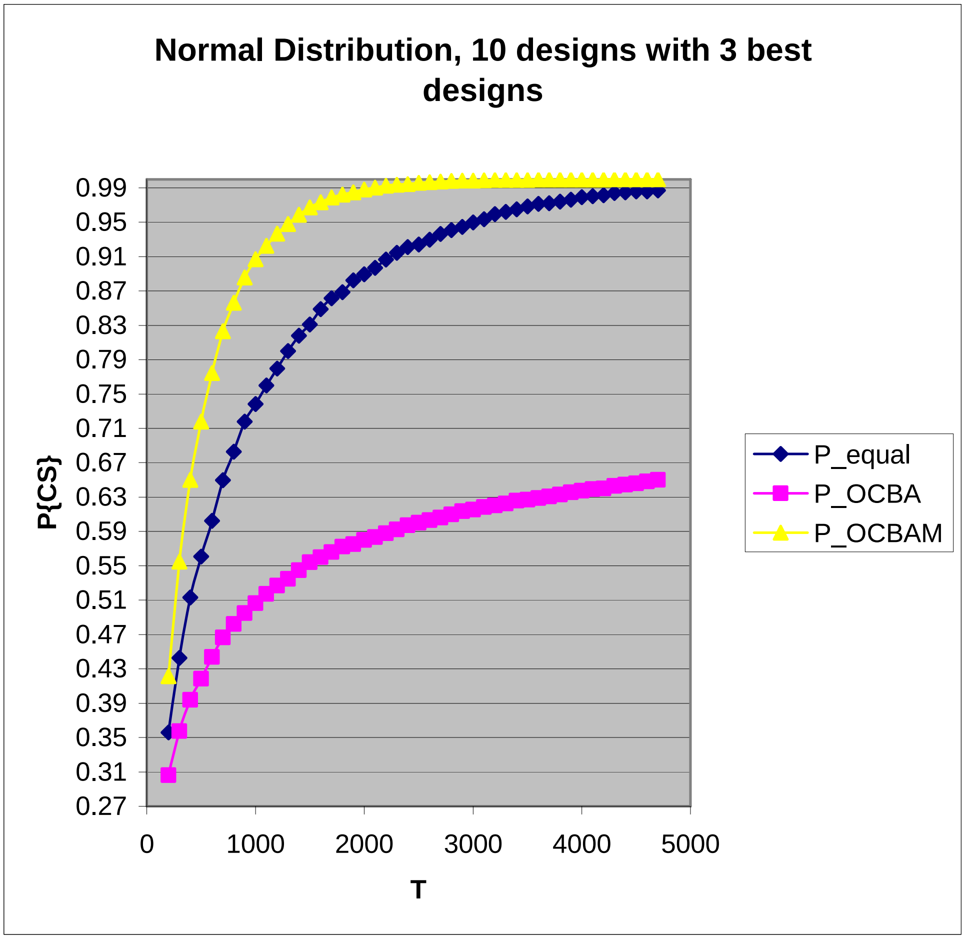
Different computing budgets are tested. Figure 3 shows OCBA doesn't perform well in this case and our revised OCBA method, OCBAM, can converge P{CS} to 1 much quicker.

Table 3. Normal distributions, 10 designs with 3 same best designs
T: simulation budget;
P_equal: P{CS} by equal allocation procedure;
P_OCBA: P{CS} by OCBA procedure.
P_OCBAM: P{CS} by OCBAM procedure.

T	P_equal	P_OCBA	P_OCBAM
200	0.356	0.306	0.4215
300	0.4429	0.3576	0.5547
400	0.513	0.394	0.6499
500	0.5609	0.4184	0.718
600	0.6025	0.4437	0.7745
700	0.6497	0.4666	0.8233
800	0.6827	0.4823	0.8563
900	0.7179	0.4949	0.8857
1000	0.7382	0.5065	0.9069
1100	0.7599	0.5173	0.9226
1200	0.7796	0.5269	0.9367
1300	0.8002	0.5347	0.9478
1400	0.8181	0.5448	0.9585
1500	0.831	0.5539	0.9673
1600	0.8489	0.5601	0.9733
1700	0.8616	0.5659	0.9788
1800	0.8686	0.5724	0.9822
1900	0.8822	0.5754	0.9848
2000	0.8894	0.58	0.9882
2100	0.897	0.5834	0.9904
2200	0.9064	0.588	0.9924
2300	0.9142	0.5924	0.9937
2400	0.9211	0.5972	0.9945
2500	0.9242	0.6002	0.996
2600	0.9298	0.6031	0.9967
2700	0.9362	0.6062	0.9973
2800	0.9408	0.6098	0.9981
2900	0.9446	0.6135	0.9984
3000	0.9499	0.6154	0.9984
3100	0.9534	0.6184	0.999
3200	0.9593	0.6202	0.9993
3300	0.9619	0.6227	0.9994

3400	0.9649	0.626	0.9994
3500	0.9684	0.627	0.9994
3600	0.9714	0.6288	0.9996
3700	0.972	0.6308	0.9997
3800	0.9741	0.6331	0.9998
3900	0.9763	0.6354	0.9998
4000	0.979	0.6375	0.9998
4100	0.9804	0.6394	0.9998
4200	0.9815	0.6401	0.9998
4300	0.9839	0.6429	0.9999
4400	0.9849	0.6446	0.9999
4500	0.9859	0.6461	0.9999
4600	0.9857	0.6482	0.9999
4700	0.987	0.6501	1

Figure 3. Normal distributions, 10 designs
P{CS} vs. T using OCBA, OCBAM and equal allocation procedure



Experiment 4, Normal Distribution, 100 designs with 5 best designs

There are 100 design alternatives. Suppose Xi ~ N $(0.4, 1^2)$, $i = 0, 1, 2, 3, 4$ and Xi ~ N $(i/10, 1^2)$, $i = 5, 6, .., 99$. We want to find 5 designs with the minimum mean. It is obvious that design 0, 1, 2, 3, 4 are the actual best designs.

Different computing budgets are tested. Figure 4 shows that P{CS}of OCBAM converges to 1 much quicker than the other two procedures.

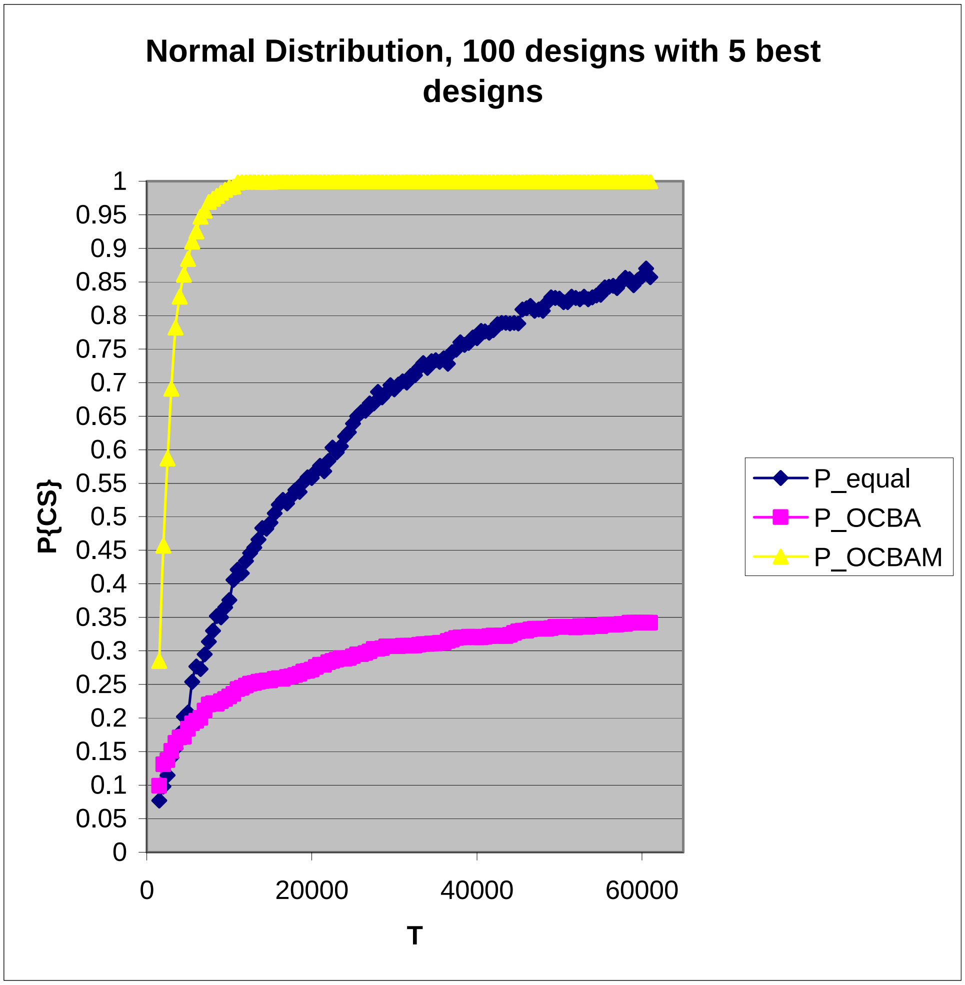
Table 4. Normal distributions, 100 designs
T: simulation budget;
P_equal: P{CS} by equal allocation procedure;
P_OCBA: P{CS} by OCBA procedure.
P_OCBAM: P{CS} by OCBAM procedure.

T	P_equal	P_OCBA	P_OCBAM
1500	0.077	0.099	0.285
2000	0.098	0.131	0.457
2500	0.115	0.137	0.587
3000	0.141	0.151	0.691
3500	0.155	0.163	0.782
4000	0.177	0.171	0.828
4500	0.202	0.172	0.861
5000	0.208	0.184	0.884
5500	0.254	0.192	0.91
6000	0.277	0.196	0.925
6500	0.273	0.2	0.947
7000	0.295	0.211	0.956
7500	0.314	0.22	0.969
8000	0.33	0.222	0.974
8500	0.352	0.221	0.978
9000	0.35	0.225	0.983
9500	0.365	0.228	0.987
10000	0.376	0.232	0.991
10500	0.406	0.236	0.992
11000	0.421	0.243	0.998
11500	0.416	0.245	0.999
12000	0.434	0.248	0.999
12500	0.446	0.251	1
13000	0.454	0.252	1
13500	0.466	0.254	0.999
14000	0.483	0.255	0.999
14500	0.482	0.256	0.999
15000	0.491	0.256	0.999
15500	0.505	0.258	1
16000	0.518	0.259	1
16500	0.525	0.258	1
17000	0.52	0.261	1
17500	0.53	0.262	1

18000	0.54	0.264	1
18500	0.537	0.266	1
19000	0.551	0.269	1
19500	0.559	0.27	1
20000	0.558	0.272	1
20500	0.568	0.276	1
21000	0.576	0.279	1
21500	0.568	0.279	1
22000	0.583	0.283	1
22500	0.603	0.285	1
23000	0.596	0.287	1
23500	0.605	0.289	1
24000	0.62	0.288	1
24500	0.626	0.289	1
25000	0.639	0.292	1
25500	0.65	0.295	1
26000	0.656	0.295	1
26500	0.658	0.297	1
27000	0.669	0.299	1
27500	0.669	0.303	1
28000	0.686	0.303	1
28500	0.678	0.304	1
29000	0.685	0.307	1
29500	0.696	0.307	1
30000	0.69	0.307	1
30500	0.697	0.307	1
31000	0.702	0.308	1
31500	0.7	0.308	1
32000	0.71	0.308	1
32500	0.711	0.308	1
33000	0.722	0.309	1
33500	0.729	0.31	1
34000	0.722	0.31	1
34500	0.732	0.311	1
35000	0.733	0.311	1
35500	0.731	0.312	1
36000	0.736	0.312	1
36500	0.728	0.315	1
37000	0.745	0.317	1
37500	0.749	0.319	1
38000	0.76	0.32	1
38500	0.756	0.32	1
39000	0.759	0.321	1
39500	0.767	0.321	1
40000	0.766	0.32	1
40500	0.777	0.32	1
41000	0.776	0.321	1
41500	0.774	0.322	1

42000	0.778	0.323	1
42500	0.787	0.322	1
43000	0.789	0.323	1
43500	0.789	0.322	1
44000	0.788	0.324	1
44500	0.789	0.327	1
45000	0.788	0.329	1
45500	0.809	0.33	1
46000	0.811	0.33	1
46500	0.814	0.332	1
47000	0.807	0.333	1
47500	0.809	0.333	1
48000	0.807	0.333	1
48500	0.819	0.333	1
49000	0.827	0.334	1
49500	0.826	0.336	1
50000	0.825	0.336	1
50500	0.82	0.336	1
51000	0.82	0.336	1
51500	0.828	0.336	1
52000	0.826	0.335	1
52500	0.824	0.337	1
53000	0.828	0.337	1
53500	0.824	0.336	1
54000	0.827	0.338	1
54500	0.83	0.338	1
55000	0.831	0.337	1
55500	0.842	0.339	1
56000	0.843	0.339	1
56500	0.844	0.339	1
57000	0.841	0.339	1
57500	0.849	0.34	1
58000	0.856	0.34	1
58500	0.854	0.342	1
59000	0.845	0.342	1
59500	0.853	0.342	1
60000	0.858	0.342	1
60500	0.87	0.342	1
61000	0.857	0.342	1

Figure 4. Normal distributions, 100 designs
P{CS} vs. T using OCBA, OCBAM and equal allocation procedure



4. Conclusions

The authors present a highly efficient procedure to identify the best design out of *k* (simulated) competing designs. The purpose of this technique is to further enhance the efficiency of ordinal optimization in simulation experiments. The objective is to maximize the simulation efficiency, expressed as the probability of correct selection within a given computing budget. Their procedure allocates replications in a way that optimally improves an asymptotic approximation to the probability of correct selection.

We used same approach to minimize the computing budget subject to a satisfactory confidence level, expressed as the probability of correct selection. As expected, the procedure is same with the procedure given by the paper.

We also consider the case that there are multiple best designs and we revise OCBA procedure to solve the problem. Next step should be to develop a method to identify the number of best designs during simulation.

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