Computing Efforts Allocation for Ordinal Optimization and Discrete Event Simulation

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***Abstract*—Ordinal optimization has emerged as an efficient technique for simulation and optimization. Exponential convergence rates can be achieved in many cases. In this paper, we present a new approach that can further enhance the efficiency of ordinal optimization. Our approach intelligently determines the optimal number of simulation replications (or samples) and significantly reduces the total simulation cost. Numerical illustrations are included. The results indicate that our approach can obtain an additional 74% computation time reduction above and beyond the reduction obtained through the use of ordinal optimization for a 10-design example.**

I. Introduction

Discrete-event systems (DES) simulation is a popular tool for analyzing systems and evaluating decision problems since real situations rarely satisfy the assumptions of analytical models. While DES simulation has many advantages for modeling complex systems, efficiency is still a significant concern when conducting simulation experiments. To obtain a good statistical estimate for a design decision, a large number of simulation replications[1] are usually required for each design alternative. This is due to the slow convergence of a performance measure estimator relative to the number of replications. The ultimate accuracy (typically measured by the width of the confidence interval) of this estimate cannot improve faster than $O(1/\sqrt{N})$, as a result of averaging i.i.d. noise over N independent simulation replications [12], [16]. Suppose we want to compare k different systems (e.g., competing designs or alternative operating policies). We conduct N simulation replications for each of the k designs. Therefore, we need kN simulation replications. Simulation results become more accurate as N increases. If the accuracy requirement is high (N is not small) and if the total number of designs in a decision problem is large (k is large), then kN can be very large, which may easily make the total simulation cost prohibitively high.

If our goal is to find the best design or a good design rather than an accurate estimate of the best performance value, it is advantageous to use ordinal comparison first proposed in [15]. Dai [9] shows that the convergence rate for ordinal optimization can be exponential. The convergence rate increases significantly under ordinal comparison since a superior design can be detected even when the value estimate is still poor. This idea has been successfully applied to several decision problems [2], [3], [7], [14], [18], [20].

While *ordinal optimization* could significantly reduce the computational cost for DES simulation, there is potential to further improve its performance by intelligently determining the number of simulation samples to be allocated among different designs. Intuitively, to ensure a high *alignment probability*, i.e., the probability that the competing designs are ranked correctly, a larger portion of the computing budget should be allocated to those designs that are critical in the process of identifying good designs. Note that the alignment probability is also called the *probability of correct selection* or $P\{\text{CS}\}$ in the simulation literature (e.g., [1] and [17]). In other words, a larger number of simulations must be conducted with those critical designs in order to reduce estimator variance. On the other hand, limited computational effort should be expended on noncritical designs that have little effect on identifying the good designs even if these noncritical designs have large variances. In doing so, less computational effort is spent on simulating noncritical designs and more computational effort is spent on simulating critical designs; hence, the overall simulation efficiency is improved. Ideally, we want to optimally choose the number of simulation samples for all designs to maximize simulation efficiency with a given computing budget. In this paper, we present a technique called *optimal computing budget allocation* (OCBA) which implements this idea.

Chen [4] formulates the procedure of allocating computational effort to competing designs as a nonlinear optimization problem. Chen *et al.* [6] apply the steepest ascent method to solve the resulting budget allocation problem. An extra computation cost is incurred to iteratively search for a solution to the budget allocation problem. In this paper, we consider a different approach. We replace the objective function with an approximation and then find an analytical solution to the approximation problem.

The paper is organized as follows: In the next section, we formulate the optimal computing budget allocation problem and discuss the major issues in solving this optimization problem. Since our approach is based on the Bayesian model, we also provide a brief discussion of that model for completeness. Section III presents an asymptotic allocation rule for OCBA. The performance of the technique is illustrated with a numerical example in Section IV. Section V concludes the paper.

II. Notations and Problem Formulation

In this section, we establish the notation used in this paper and then formulate the computing budget allocation problem. Denote by the following.

- k total number of designs;
- X_{ij} jth i.i.d. sample of the performance measure from design i;
- N_i number of simulation replications for design i;
- $\boldsymbol{X}_i$ vector representing the simulation output for design i; $\boldsymbol{X}_i = \{X_{ij}: j = 1, 2, \cdots, N_i\}$;
- $\overline{\mu}_i$ sample mean performance measure for design i, $\overline{\mu}_i = (1/N_i)\Sigma_{j=1}^{N_i} X_{ij}$;
- μ_i mean performance measure; $\mu_i = E(X_{ij})$;
- σ_i^2 variance for design, i;
- b design having the smallest sample mean performance measure, i.e., $\overline{\mu}_b \leq \min_i \overline{\mu}_i$;
- s design having the second smallest sample mean performance measure, i.e., $\overline{\mu}_b \leq \overline{\mu}_s \leq \min_{i\neq b} \overline{\mu}_i$;
- $\delta_{j,i}$ $\equiv \overline{\mu}_j - \overline{\mu}_i$.

Note that when N_i is large, $\overline{\mu}_i$ can be a good approximation to μ_i, since, according to the law of large numbers, $P\{\lim_{N_i\to\infty} \overline{\mu}_i = \mu_i\} = 1$. Under the framework of ordinal optimization, we will have only a small number of simulation replications or simulation samples. We are therefore concerned with the alignment probability, the probability that the competing designs are ranked correctly and that at least a good design is identified. While $\overline{\mu}_i$ converges to μ_i slowly, the alignment probability of ordinal comparison converges to 1.0 exponentially if the moment generating function of X_{ij} is finite [9]. Taking advantage of exponential convergence, we intend to further improve the alignment probability or simulation quality using the same computing budget.

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[1]We consider terminating (finite-horizon) simulations in this paper. Our approach is equally applicable to steady-state simulations where we need N independent samples rather than N independent simulation replications.

TABLE I
$P\{\text{CS}\}$ AND AVERAGE *APCS* BY APPLYING OO ALONE VERSUS OO+OCBA

Desired $P\{\text{CS}\}$	OO Alone			OO + OCBA			Time savings using OCBA
	T	$P\{\text{CS}\}$	Av. *APCS*	T	$P\{\text{CS}\}$	Av. *APCS*	
95%	2000	95.62%	88.75%	700	96.57%	88.98%	65%
98%	2900	98.07%	92.82%	900	98.06%	92.48%	69%
98.5%	3500	98.71%	94.52%	1100	99.08%	94.88%	69%
99.5%	5000	99.63%	97.17%	1300	99.58%	96.46%	74%

If a simulation is performed on a sequential computer and the difference of computation costs of simulating different designs is negligible, the total computation cost can be approximated by $N_1+N_2+\cdots+N_k$. Our goal is to choose $N_1, N_2, \cdots$, and N_k, such that the alignment probability is maximized, subject to a limited computing budget T

$$\max_{N_1,\cdots,N_k} \{\text{Alignment Probability}\}$$
$$\text{s.t. } N_1 + N_2 + \cdots + N_k = T. \tag{1}$$

To solve the budget allocation problem in (1), we must be able to estimate the alignment probability (or the probability of correct selection or $P\{\text{CS}\}$) easily. There exists a large literature on assessing $P\{\text{CS}\}$ based on classical statistical model. Goldsman and Nelson [13] provide an excellent survey on available approaches to ranking, selection, and multiple comparisons. However, most of these approaches are only suitable for problems with a small number of designs (e.g., [13] suggest 2–20 designs). Using a Bayesian model, Chen [5] introduces an estimation technique to quantify the confidence level for ordinal comparison when the number of designs is large. In addition to the confidence probability, this approach also provides sensitivity information for each algorithm, which is useful when solving the allocation problem in (1).

Following the Bayesian model in [5], we assume that the simulation output, X_{ij}, has a normal distribution with mean μ_i and known σ_i^2. After the simulation is performed, *a posterior* distribution of μ_i, $p(\mu_i|\boldsymbol{X}_i)$, can be constructed based on two pieces of information: i) the prior knowledge on the system's performance and ii) the simulation output, the observed performance. Thus, the probability of correctly selecting the best design can then be defined by

$$\begin{aligned} P\{\text{CS}\} &= P\{\text{design } b \text{ is actually the best design}\} \\ &= P\{\mu_b < \mu_i, i \neq b \,|\, \boldsymbol{X}_i, i = 1, 2, \cdots, k\}. \end{aligned} \tag{2}$$

To simplify the notation used, we rewrite (2) as $P\{\hat{\mu}_b < \hat{\mu}_i, i \neq b\}$, where $\hat{\mu}_i$ denotes the random variable whose probability distribution is the posterior distribution for design i. Further assume that the unknown mean μ_i has the conjugate normal prior distribution $N(\mu_0, \nu_0^2)$. While the prior distribution is not changed as the simulation proceeds, the posterior distribution is updated based on the simulation output. We assume that the performance of any design is totally unknown before conducting the simulation. In that case, DeGroot [10] suggests a procedure whereby a prior distribution is found by taking the parameter of the conjugate prior distribution to some limiting value. Chick [8] shows that the posterior distribution of μ_i is then given by

$$\hat{\mu}_i \sim N\left(\frac{1}{N_i}\sum_{j=1}^{N_i} X_{ij},\ \frac{\sigma_i^2}{N_i}\right).$$

After the simulation is performed, $\overline{\mu}_i$ can be calculated, σ_i^2 can be approximated by the sample variance, and then $P\{\text{CS}\}$ can be estimated using a Monte Carlo simulation. However, estimating $P\{\text{CS}\}$ via Monte Carlo simulation is time consuming. Since the purpose of budget allocation is to improve simulation efficiency, we need a relatively fast and inexpensive way of estimating $P\{\text{CS}\}$ *within* the budget allocation procedure. Efficiency is more crucial than estimation accuracy in this setting.

Let Y_i be a random variable. According to the Bonferroni inequality, $P\{\cap_{i=1}^k (Y_i < 0)\} \geq 1 - \Sigma_{i=1}^k [1 - P\{(Y_i < 0)\}]$. We replace Y_i by $(\hat{\mu}_b - \hat{\mu}_i)$, to provide a lower bound for the probability of correct selection. That is

$$\begin{aligned} P\{\text{CS}\} &= P\left\{\bigcup_{\substack{i=1 \\ i\neq b}}^{k} (\hat{\mu}_b - \hat{\mu}_i < 0)\right\} \\ &\geq 1 - \sum_{i=1, i\neq b}^{k} [1 - P\{\hat{\mu}_b - \hat{\mu}_i < 0\}] \\ &= 1 - \sum_{i=1, i\neq b}^{k} P\{\hat{\mu}_b > \hat{\mu}_i\} \\ &= APCS. \end{aligned}$$

We refer to this lower bound of the correct selection probability as the approximate probability of correct selection (*APCS*). *APCS* can be computed very easily and quickly; no extra Monte Carlo simulation is needed. Numerical testing in [5] shows that *APCS* provides a good approximation to $P\{\text{CS}\}$. The approximation property of *APCS* can also be seen from Table I in Section IV. We therefore use *APCS* to approximate $P\{\text{CS}\}$ as the simulation experiment proceeds. More specifically, we consider the following problem:

$$\max_{N_1,\cdots,N_k} 1 - \sum_{i=1, i\neq b}^{k} P\{\hat{\mu}_b > \hat{\mu}_i\}$$
$$\text{s.t. } \sum_{i=1}^{k} N_i = T \text{ and } N_i \geq 0. \tag{3}$$

In the next section, an asymptotic allocation rule with respect to the number of simulation replications, N_i, will be presented.

III. An Asymptotic Allocation Rule

First, we assume the decision variables, N_i's, are continuous. Second, our strategy is to tentatively ignore all nonnegativity constraints; all N_i's can therefore assume any real value. Before the end of this section, we show how all N_i's automatically become positive. To further facilitate the computations, we intend to find an asymptotic allocation rule. Namely, we consider the case that $T \to +\infty$. While it is impossible to have an infinite computing budget in real life, our allocation rule provides a means for allocating simulation budget in a way that the efficiency can be significantly improved, as we will demonstrate in numerical testing later. Based on this idea, we first consider the following:

$$\max_{N_1,\cdots,N_k} 1 - \sum_{i=1, i\neq b}^{k} P\{\hat{\mu}_b > \hat{\mu}_i\}$$
$$\text{s.t. } \sum_{i=1}^{k} N_i = T. \tag{4}$$

The major issue for solving (4) is the estimation of the gradient information of $\Sigma_{i=1,i\neq b}^{k} P\{\hat{\mu}_b > \hat{\mu}_i\}$. This is not easy because of the lack of a closed-form formula for $\Sigma_{i=1,i\neq b}^{k} P\{\hat{\mu}_b > \hat{\mu}_i\}$. For that reason, we introduce Chernoff bounds in Lemma 1 to provide the necessary information for estimating the gradient.

Lemma 1: Suppose the random variable Y is distributed according to $N(\delta_{b,i}, (\sigma_b^2/N_b) + (\sigma_i^2/N_i))$, where

$$\delta_{b,i} = (\overline{\mu}_b - \overline{\mu}_i) < 0.$$

Then

$$P\{Y > 0\} \leq \exp\left(\frac{-\delta_{b,i}^2}{2\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}\right)}\right).$$

Proof: See [19].

With this lemma

$$\begin{aligned} APCS &= 1 - \sum_{i=1,i\neq b}^{k} P\{\hat{\mu}_b - \hat{\mu}_i > 0\} \\ &\geq 1 - \sum_{i=1,i\neq b}^{k} \exp\left(\frac{-\delta_{b,i}^2}{2\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}\right)}\right) \\ &= EAPCS \text{ (estimated approximate probability of correct selection).} \end{aligned}$$

Note that as $N_b \to \infty$ and $N_i \to \infty$, $\exp((-\delta_{b,i}^2/2)(\sigma_b^2/N_b + \sigma_i^2/N_i)^{-1}) \to 0$ and $P\{\lim_{N_i\to\infty} EAPCS = APCS$, for $i = 1, 2, \cdots, b, \cdots, k\} = 1$. Thus, the following optimal budget allocation can be considered as an approximation to (4):

$$\begin{aligned} &\max_{N_i,\cdots,N_k} 1 - \sum_{i=1,i\neq b}^{k} \exp\left(\frac{-\delta_{b,i}^2}{2\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}\right)}\right) \\ &\text{s.t. } \sum_{i=1}^{k} N_i = T. \end{aligned} \tag{5}$$

Then let F be the Lagrangian relaxation of (5)

$$F = \left[1 - \sum_{i=1,i\neq b}^{k} \exp\left(-\frac{\delta_{b,i}^2}{2\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}\right)}\right)\right] - \lambda\left(\sum_{i=1}^{k} N_i - T\right).$$

Furthermore, the Karush–Kuhn–Tucker (KKT) conditions of this problem can be stated as follows:

$$\begin{aligned} \frac{\partial F}{\partial N_i} &= \exp\left(\frac{-\delta_{b,i}^2}{2\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}\right)}\right) \cdot \frac{\delta_{b,i}^2 \sigma_i^2/N_i^2}{\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}\right)^2} \\ &\quad - \lambda = 0, \qquad \text{for } i = 1, 2, \cdots, k, \text{ and } i \neq b \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{\partial F}{\partial N_b} &= \sum_{\substack{i=1\\ i\neq b}}^{k} \exp\left(\frac{-\delta_{b,i}^2}{2\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}\right)}\right) \frac{\delta_{b,i}^2 \sigma_b^2/N_b^2}{\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}\right)^2} - \lambda = 0 \\ &\lambda\left(\sum_{i=1}^{k} N_i - T\right) = 0, \qquad \text{and } \lambda \geq 0. \end{aligned} \tag{7}$$

We now examine the relationship between N_i and N_s for $i = 1, 2, \cdots, k$, and $i \neq b \neq s$. From (6)

$$\begin{aligned} &\exp\left(\frac{-\delta_{b,i}^2}{2\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}\right)}\right) \cdot \frac{\delta_{b,i}^2 \sigma_i^2/N_i^2}{\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}\right)^2} \\ &\quad = \exp\left(\frac{-\delta_{b,s}^2}{2\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_s^2}{N_s}\right)}\right) \cdot \frac{\delta_{b,s}^2 \sigma_s^2/N_s^2}{\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_s^2}{N_s}\right)^2}. \end{aligned}$$

To maximize $\Sigma_{i=1,i\neq b}^{k} P\{\hat{\mu}_b < \hat{\mu}_i\}$, N_b should be increased relative to N_i (this is indeed the case we observe from the actual simulation experiments), for $i = 1, 2, \cdots, k$, and $i \neq b$. Therefore, we assume that $N_b \gg N_i$, which enables us to simplify the above equation as

$$\exp\left(\frac{-\delta_{b,i}^2}{2\left(\frac{\sigma_i^2}{N_i}\right)}\right) \cdot \frac{\delta_{b,i}^2 \sigma_i^2/N_i^2}{\left(\frac{\sigma_i^2}{N_i}\right)^2} \approx \exp\left(\frac{-\delta_{b,s}^2}{2\left(\frac{\sigma_s^2}{N_s}\right)}\right) \cdot \frac{\delta_{b,s}^2 \sigma_s^2/N_s^2}{\left(\frac{\sigma_s^2}{N_s}\right)^2}.$$

Then

$$\exp\left(\frac{-\delta_{b,i}^2}{2\left(\frac{\sigma_i^2}{N_i}\right)}\right) \cdot \frac{\delta_{b,i}^2}{\sigma_i^2} \approx \exp\left(\frac{-\delta_{b,s}^2}{2\left(\frac{\sigma_s^2}{N_s}\right)}\right) \cdot \frac{\delta_{b,s}^2}{\sigma_s^2}.$$

On rearranging terms, the above equation becomes

$$\exp\left(\frac{1}{2}\left(\frac{\delta_{b,s}^2}{\frac{\sigma_s^2}{N_s}} - \frac{\delta_{b,i}^2}{\frac{\sigma_i^2}{N_i}}\right)\right) \approx \frac{\delta_{b,s}^2}{\sigma_s^2}\,\frac{\sigma_i^2}{\delta_{b,i}^2}.$$

Taking the natural log on both sides, we have

$$\frac{\delta_{b,s}^2}{\sigma_s^2} N_s \approx \frac{\delta_{b,i}^2}{\sigma_i^2} N_i + 2\log\left(\frac{\delta_{b,s}^2}{\sigma_s^2}\,\frac{\sigma_i^2}{\delta_{b,i}^2}\right).$$

N_i and N_s will increase as simulation proceeds, which implies that the value of the first two terms of the above equation is much greater than the log term. This implies

$$\frac{\delta_{b,s}^2}{\sigma_s^2} N_s \approx \frac{\delta_{b,i}^2}{\sigma_i^2} N_i \qquad \text{as } N_i, N_s \to \infty.$$

Therefore, we obtain the ratio between N_i and N_s as

$$\frac{N_i}{N_s} \approx \left(\frac{\sigma_i/\delta_{b,i}}{\sigma_s/\delta_{b,s}}\right)^2, \qquad \text{for } i = 1, 2, \cdots, k, \text{ and } i \neq b \neq s. \tag{8}$$

We further investigate the ratio between N_b and N_s. From (6) and (7)

$$\begin{aligned} &\exp\left(\frac{-\delta_{b,s}^2}{2\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_s^2}{N_s}\right)}\right) \cdot \frac{\delta_{b,s}^2 \sigma_s^2/N_s^2}{\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_s^2}{N_s}\right)^2} \\ &\quad = \sum_{\substack{i=1\\ i\neq b}}^{k} \exp\left(\frac{-\delta_{b,i}^2}{2\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}\right)}\right) \cdot \frac{\delta_{b,i}^2 \sigma_b^2/N_b^2}{\left(\frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}\right)^2}. \end{aligned}$$

Again, we can simplify the above equation based on the assumption of $N_b \gg N_i$ for $i = 1, 2, \cdots, k$, and $i \neq b$, then

$$\exp\left(\frac{-\delta_{b,s}^2}{2\left(\frac{\sigma_s^2}{N_s}\right)}\right) \cdot \frac{\delta_{b,s}^2 \sigma_s^2/N_s^2}{\left(\frac{\sigma_s^2}{N_s}\right)^2} \approx \sum_{\substack{i=1\\ i\neq b}}^{k} \left[\exp\left(\frac{-\delta_{b,i}^2}{2\left(\frac{\sigma_i^2}{N_i}\right)}\right) \cdot \frac{\delta_{b,i}^2 \sigma_b^2/N_b^2}{\left(\frac{\sigma_i^2}{N_i}\right)^2}\right]. \tag{9}$$

The right-hand side of (9) becomes

$$\frac{\sigma_b^2}{N_b^2} \exp\left(\frac{-\delta_{b,s}^2}{2\left(\frac{\sigma_s^2}{N_s}\right)}\right) \cdot \left[\frac{\delta_{b,s}^4}{\left(\frac{\sigma_s^2}{N_s}\right)^2} \sum_{\substack{i=1\\ i\neq b}}^{k} \left(\frac{1}{\delta_{b,i}^2}\right)\right]$$

after N_i is replaced by $N_s \sigma_i^2 \delta_{b,s}^2 / \sigma_s^2 \delta_{b,i}^2$, using the results in (8).

Furthermore, canceling

$$\exp\left(\frac{-\delta_{b,s}^2}{2}\left(\frac{\sigma_s^2}{N_s}\right)^{-1}\right)$$

and $\delta_{b,s}^2(\sigma_s^2/N_s)^{-2}$ on both sides, (9) can be simplified as follows:

$$\frac{N_b}{N_s} \approx \frac{\sigma_b}{\sigma_s}\left[\sum_{\substack{i=1\\ i\neq b}}^{k}\left(\frac{\delta_{b,s}^2}{\delta_{b,i}^2}\right)\right]^{1/2}. \tag{10}$$

We further observe that all N_i's have the same sign from (8) and (10). Since $\Sigma_{i=1}^{k} N_i = T$ and $T > 0$, it implies that all N_i's > 0, where $i = 1, 2, \cdots, k$.

In conclusion, if a solution satisfies (8) and (10), then K–K–T conditions must hold. According to the K–K–T sufficient condition, this solution is a locally optimal solution to (5). On the other hand, when T is large, the number of simulation replications of each design will be large, so that *EAPCS* approaches *APCS* as in (5). We therefore have the following result.

Theorem 1: Given a total number of simulation replications T to be allocated to k competing designs whose performance is depicted by random variables with means $\mu_1, \mu_2, \cdots, \mu_k$, and finite variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2$, respectively, as $T \to \infty$, the *APCS* can be maximized when

1) $$\frac{N_b}{N_s} \to \frac{\sigma_b}{\sigma_s}\left[\sum_{\substack{i=1\\ i\neq b}}^{k}\left(\frac{\delta_{b,s}^2}{\delta_{b,i}^2}\right)\right]^{1/2}$$

2) $$\frac{N_i}{N_s} \to \left(\frac{\sigma_i/\delta_{b,i}}{\sigma_s/\delta_{b,s}}\right)^2, \qquad \text{for } i = 1, \cdots, k \text{ and } i \neq s \neq b$$

where N_i is the number of replications allocated to design i, $\delta_{b,i} = \overline{\mu}_b - \overline{\mu}_i$ and $\overline{\mu}_b \leq \overline{\mu}_s \leq \min_{i\neq b\neq s} \overline{\mu}_i$.

Given the complexity of $P\{\text{CS}\}$, we take several steps of approximations in order to obtain Theorem 1. While it is difficult to estimate the impact of approximations, we will show that using Theorem 1 can indeed significantly reduce the simulation time on a numerical example in the next section. From part (1) of the theorem, it can be seen that N_b has a summation property of all other N_i's. Thus, the assumption we made earlier that $N_b \gg N_i$ is valid, if the total number of designs is not small.

On the other hand, in an opposite case that $k = 2$, N_b is usually not much bigger than N_i. However, Theorem 1 gives an allocation that is identical to a well-known optimal solution in the following remark.

Remark 1: In the case of $k = 2$ and $b = 1$, Theorem 1 yields

$$\frac{N_1}{N_2} = \frac{\sigma_1}{\sigma_2}\left[\frac{\delta_{1,2}^2}{\delta_{1,2}^2}\right]^{1/2} = \frac{\sigma_1}{\sigma_2}.$$

This evaluated result is identical to the well-known optimal allocation solution for $k = 2$.

It is worthy to point out that when the top-two designs are very close, most simulation budget will be allocated to these two designs since it is difficult to distinguish these two designs. In such a case, the budget allocation for these two designs will be approximately governed by Remark 1 because the allocation to other designs is almost negligible.

With Theorem 1, we now present a cost-effective sequential approach based on OCBA to select the best design from k alternatives with a given computing budget. The main idea of the heuristic algorithm for the solution to OCBA is briefly stated as follows: Initially n_0 simulation replications for each of the k design are conducted to get some information about the performance of each design during the first stage. As the simulation proceeds, the sample means and sample variances of each design are computed from the data already collected up to that stage. According to this observed simulation output, an incremental computing budget, Δ, is allocated based on Theorem 1 at each stage. Ideally each new replication should bring us closer to the optimal solution. This procedure is continued until the total budget T is exhausted. The algorithm is summarized as follows.

A Sequential Algorithm for Optimal Computing Budget Allocation (OCBA):

Step 0: Perform n_0 simulation replications for all designs; $l \leftarrow 0$; $N_1^l = N_2^l = \cdots = N_k^l = n_0$.

Step 1: If $\Sigma_{i=1}^{k} N_i^l \geq T$, stop.

Step 2: Increase the computing budget (i.e., number of additional simulations) by Δ and compute the new budget allocation, $N_1^{l+1}, N_2^{l+1}, \cdots, N_k^{l+1}$, using Theorem 1.

Step 3: Perform additional $\max(0, N_i^{l+1} - N_i^l)$ simulations for design i, $i = 1, \cdots, k$. $l \leftarrow l + 1$. Go to Step 1.

IV. Numerical Testing

This test case is a G/G/1 queue in which the objective is to select the design with the minimum expected waiting time over a set of ten competing designs $(k = 10)$. All designs have the same interarrival time that is uniformly distributed over [0.1, 1.9]. Service time in design i is uniformly distributed over $[0.1, 1.3 + 0.05i]$, $i = 1, 2, \cdots, 10$. We want to find a design with the minimum average waiting time for customers served within the first ten time units (terminating simulation). Since a higher service rate results in shorter waiting times, design 1 is the *actual* best design. In the numerical experiment, we also compare the convergence of $P\{\text{CS}\}$ for different setting.

We have $n_0 = 10\Delta = 5$. Furthermore, 10 000 independent experiments are performed to estimate $P\{\text{CS}\}$. In all the numerical illustrations, we estimate $P\{\text{CS}\}$ by counting the number of times we successfully find the true best design (design 1 in this example) in those 10 000 independent experiments. $P\{\text{CS}\}$ is then obtained by dividing this number by 10 000, representing the correct selection frequency. Since *APCS* can be estimated easily, we also calculate the average *APCS* over these 10 000 experiments as a reference.

To test the benefit of our OCBA technique for this problem, different computing budgets are tested. Table I shows the test results using ordinal optimization (OO) alone (the computing budget is distributed equally among all designs) and using OO+OCBA. We can see that based on the idea of ordinal optimization, with a modest total budget of 2000 replications, the probability of correctly selecting the best design is already higher than 95%. This computation cost can be further reduced to 700 replications if OCBA is applied.

TABLE II
AVERAGES OF N_1, N_4, N_7, AND N_{10} OVER THE 10 000 INDEPENDENT EXPERIMENTS USING OO+OCBA FOR DIFFERENT COMPUTING BUDGETS

T	100	400	700	1000	1300
N_1	10.0	186.2	346.7	498.1	651.2
N_2	10.0	86.9	156.5	228.5	289.4
N_4	10.0	21.6	39.5	57.5	72.5
N_7	10.0	10.0	12.8	18.6	23.9
N_{10}	10.0	10.0	10.2	10.6	11.2

We also see that using the OCBA techniques can reduce computation cost by as much as 74% in this 10-design example. Moreover, the savings factor increases as $P\{\text{CS}\}$ increases. We obtain very significant time savings for achieving the same level of $P\{\text{CS}\}$. The reason is because when $P\{\text{CS}\}$ increases, we have more flexibility to manipulate the budget allocation.

In order to show how the computing budget is allocated to different designs as T increases, we calculate the averages of N_i for all i at different T over these 10 000 experiments. The results for designs 1, 2, 4, 7, and 10 are shown in Table II. When $T = 100$, all designs are equally simulated to get some preliminary information. As T increases, more computing budget is allocated to good designs (particularly design 1), since they are more critical in the process of identifying the best design.

V. CONCLUSIONS

In this paper we introduce an optimal computing budget allocation technique for the selection of the best design out of k (simulated) competing designs. An asymptotic allocation rule is presented. This technique can further enhance the simulation efficiency of ordinal optimization. The objective is to maximize the simulation efficiency, expressed as the probability of correct selection within a given computing budget. Comparisons with the crude ordinal optimization show that our approach can achieve an extra 74% computation time reduction on for a 10-design G/G/1 example.

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Optimal Adaptive Control for Estimation of Parameters of ARX Models

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***Abstract*—The objective of this paper is to find an adaptive control strategy which would enable us to estimate the parameters of the ARX model as accurately as possible while keeping the output of the system below a specified level of variability. It turns out that this optimal experiment design problem can be solved by using the Åström–Wittenmark self tuning tracker with a specific choice of the reference signal.**

***Index Terms*—Adaptive control, ARX, optimal experiment design.**

I. INTRODUCTION

This paper is devoted to an optimal adaptive experiment design for estimation of parameters of the so-called autoregressive with exogenous input (ARX) models

$$y_t = -\sum_{i=1}^{N} a_i y_{t-i} + u_{t-1} + \xi_t, \qquad t = 1, 2, \cdots \tag{1.1}$$

where y_t, $u_{t-1} \in \mathbb{R}$, $t = 1, 2, \cdots$, are system output and input, respectively; ξ_t, $t = 1, 2, \cdots$, are i.i.d. disturbances with normal distri-

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