Promoting Decimal Number Sense and Representational Fluency

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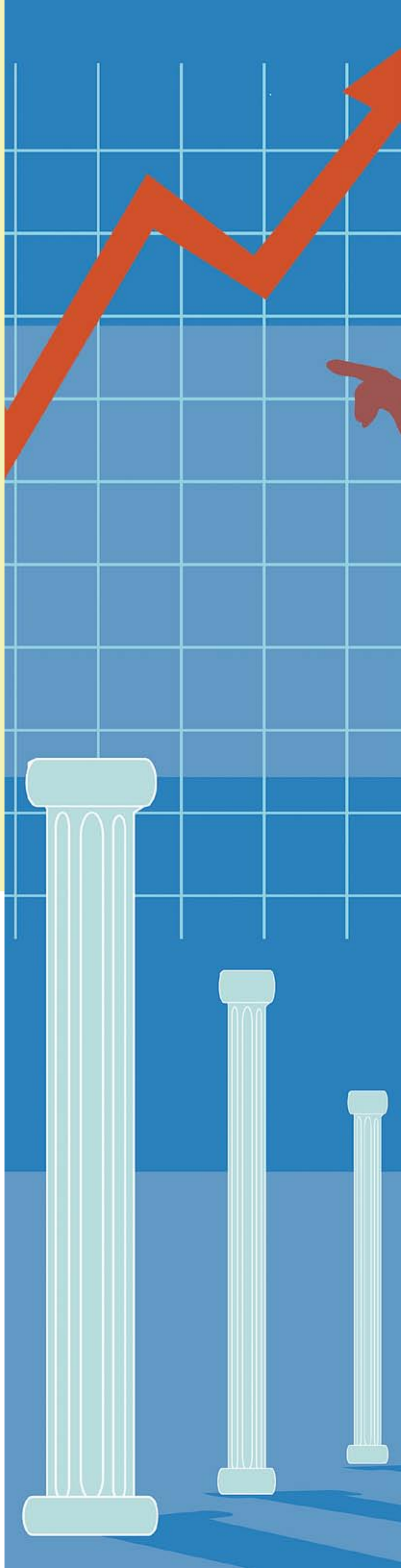

The abstract nature of mathematics requires the communication of mathematical ideas through multiple representations, such as words, symbols, pictures, objects, or actions. Building representational fluency involves using mathematical representations flexibly and being able to interpret and translate among these different models and mathematical concepts. This article shares a collaborative lesson study experience in planning and teaching a unit on decimals. Participants included fifth- and sixth-grade teachers and lesson study facilitators, including a university mathematics educator, a doctoral student, and a school mathematics specialist. The lesson was taught in a fifth-grade class with a high population of English language learners (ELL) and special needs students. The overarching goal of the lesson study was to develop students' representational fluency

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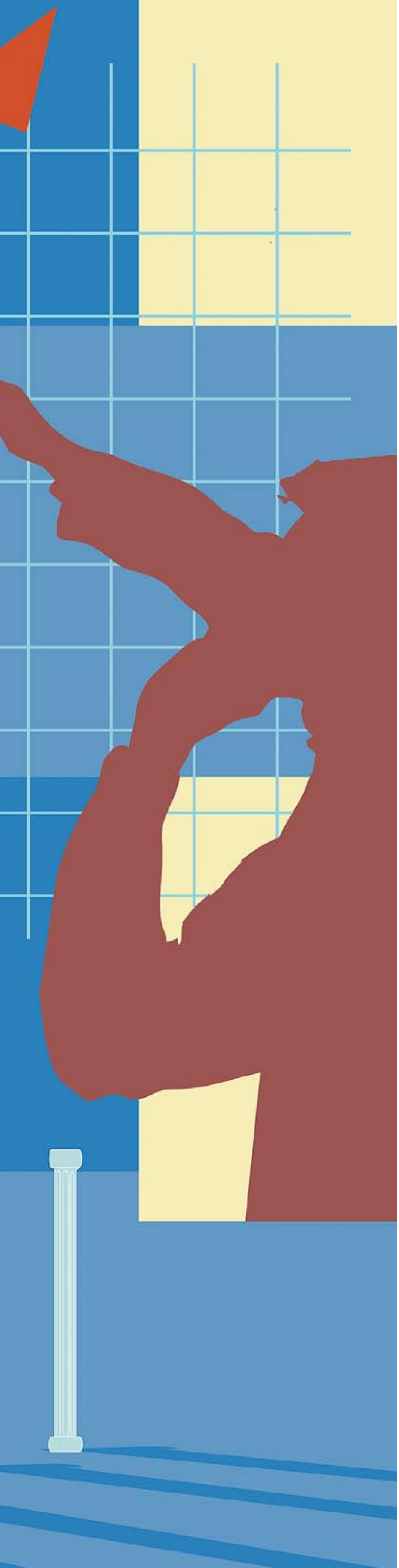
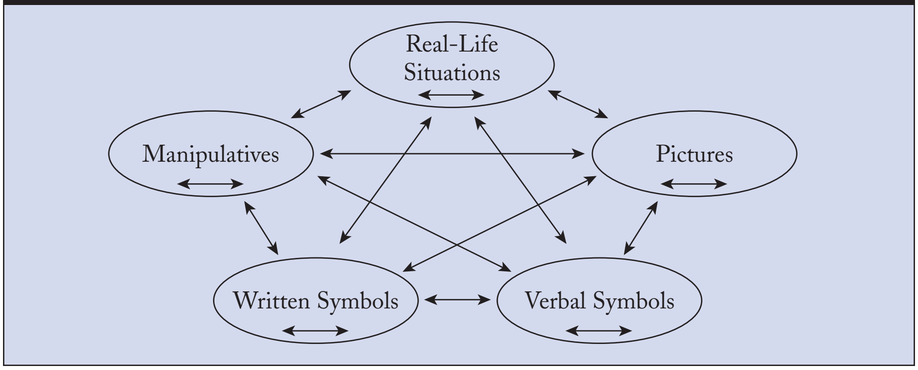


Fig. 1 Five distinct types of representations (Lesh et al. 2003)



and mathematical proficiency with decimals. While working with teachers, the lesson study facilitators shared related research on representations and the importance of selecting and evaluating effective mathematical models to give the diverse population access to decimal concepts. The lesson study facilitators' goal was to heighten teachers' awareness of the importance of multiple representations and introduce a planning process that allows teachers to select models in a thoughtful and critical way that would facilitate the teaching and learning of a mathematics concept.

UNDERSTANDING THE IMPORTANCE OF REPRESENTATIONAL FLUENCY

Representational fluency, the ability to use multiple representations and translate among these models, is key to the process of building students' mathematical understanding (Fennell and Rowan 2001; Goldin and Shteingold 2001; Lamon 2001). The Lesh Translation Model (see **fig. 1**) highlights the importance of students being able to represent mathematical ideas in multiple ways, including with manipulatives, real-life situations, pictures, verbal symbols, and written symbols (Lesh et al. 2003). A student's ability to translate among different representations allows teachers to assess conceptual understanding. Some of the ways to demonstrate translation among representations in mathematics include asking students to restate a problem in their own words, drawing a diagram to illustrate the problem, or acting it out.

In teaching and learning, representations can play a dual role, acting as both instructional tools and learning tools. As Lamon (2001) states, representations can be "both presentational models (used by adults in instruction) and representational models (produced by students in learning), which can play significant roles in instruction and its outcomes" (p. 146). Another way to think about representations is that they allow for construction of knowledge from "models of thinking to models for thinking" (Gravemeijer 1999). *Principles and Standards for School Mathematics* (NCTM 2000) emphasizes that representations serve as tools for communicating, justifying, sense making, and connecting ideas by stating, "Representations allow students to communicate mathematical approaches, arguments, and understanding to themselves and to others. They allow students to recognize connections among related concepts and apply mathematics to realistic problems" (p. 67). In this lesson study, teachers learned the importance of representational fluency on mathematics thinking and learning as they collaborated on the lesson planning as well as the teaching and reflecting process.

THE LESSON STUDY ON DECIMAL NUMBER SENSE

The lesson study took place in a Title 1 school close to a major metropolitan area. The school of approximately 600 students contained 51 percent Hispanic, 24 percent Asian, 16 percent Caucasian, 3 percent African American, and 4 percent other. The teacher group included two fifth-grade teachers, two sixth-grade teachers, and two special education teachers.

This lesson study process involved three phases: (1) the collaborative planning phase, in which the teacher group planned the lesson after defining the overarching goal and the important mathematics; (2) the teaching and observation phase, in which one teacher taught the focus lesson and the others observed using a predetermined observation form; and (3) the debriefing phase, in which teachers reflected on the lesson design, representations, student engagement, and evidence of learning, and discussed future steps. Some of the guiding questions crucial to the planning, teaching, learning, observing, and reflecting processes were the following:

- What is the important mathematical understanding that students need, and how do we assess evidence of their learning?
- What different representations will give students access to this concept?
- What conceptual supports and instructional strategies can best support our students' learning? How will we respond when students have difficulty?

PHASE 1: COLLABORATIVE PLANNING

Several days before teaching the research lesson, teachers were introduced to the lesson study process and participated in collaborative planning. Before the meeting, they had agreed on specific lesson objectives based on their state standards. Student were to estimate and find the sum and difference of decimals using concrete materials, drawings, or pictures, and mathematical symbols. To begin, teachers constructed a mathematics knowledge map that outlined the key components of the mathematics concept and the interrelated concepts that were prerequisite and future knowledge building blocks. In addition, they identified effective representations or models to teach that concept (see **fig. 2**).

Fig. 2 Mathematical knowledge map, with related decimal and fractions concepts and representations

During this discussion, a number of related concepts emerged, and many of the teachers discussed the need for students to be able to relate fractions and decimals through different representations. Teachers also emphasized the importance of students understanding (1) the structure of the base-ten system, a positional numeral system in which value is determined by the positions for units, tens, and hundreds and to the right of the decimal, tenths, hundredths, and thousandths; and (2) the position of each digit conveying the multiplier (a power of ten), in which each position has a value ten times that of the position to its right. Once the teachers had an opportunity to review all the concepts on their knowledge map, they decided to focus on decimal addition (through the hundredths place) as the topic for the lesson. Both fifth- and sixth-grade teachers remarked that students still struggled with decimal addition and estimating and understanding the magnitude of decimal numbers.

Next, teachers discussed all the possible representations and learning activities that they had employed in the past. Teachers explored several forms of representations: a place-value chart; decimal grids; base-ten blocks; a meterstick; and money and technology applets that related fractions, decimals, and percents. The challenge for teachers is finding the appropriate representations to teach a specific mathematical concept that students can use to build understanding. To give teachers a guideline for evaluating effective representations, we used the selection criteria created by the

National Research Council (2001), which include five characteristics to consider when selecting representations to communicate mathematical ideas:

1. Transparency: How easily can the idea be seen through the representation?
2. Efficiency: Does the representation support efficient communication and use?
3. Generality: Does the representation apply to broad classes of objects or concepts?
4. Clarity: Is the representation unambiguous and easy to use?
5. Precision: How close is the representation to the exact value? (NRC 2001, pp. 99–102).

An example of generality of representations when considering teaching place value and the base-ten system is to consider the often-used representation of base-ten blocks. The base-ten model can be applied to both whole numbers and decimal models. Since representational fluency impacts students' flexibility with their mathematical understanding, teachers' awareness of effective representations has implications for their instructional decisions and teaching practices.

After considering the effectiveness of multiple representations, teachers decided to use the decimal grids and the decimal place-value chart as the primary representations for the lesson. The game "decimal draw" (Burnett and Tickles 2006) was chosen and adapted for the fifth graders who were to be taught. This game was modified to include a place-value mat and decimal grids in addition to a set of movable digits, 0 through 9. The object of the game was for students to form two decimal addends, with the goal of getting as close to 1 as possible. During the discussion, it was decided that a task sheet would be a good way for students to record the results of the game and to provide a means of assessment for the teachers. One goal when creating this task sheet was to include several representations, such as writing the numbers, shading in the decimal grid, and using manipulatives. The final task sheet included place-value charts (with the ones, decimal, tenths, and hundredths identified), as well as 10 × 10 grids so students could shade in these values. This design supported our ELL students and aligned with a schoolwide initiative to build academic vocabulary (Marzano, Pickering, and Pollock 2001) through nonlinguistic representations. The design of the task sheet gave students opportunities to use different representations to explore the concept of decimal addition. During this planning session, it was also decided that students were to be given cards containing printed digits so that they could move the digits around to create numbers. We felt this would be beneficial to all students, especially the tactile learners.

PHASE 2: TEACHING WITH MULTIPLE REPRESENTATIONS

The lesson began by accessing students' prior knowledge about decimals and fractions through a class discussion. Before the lesson, the teacher had put a place-value chart on the board as well as the words *decimal, fraction, ones, decimal point, tenths*, and *hundredths*. These visual vocabulary cards were prepared to give ELL and special needs learners access to the mathematical activity. To access prior knowledge, the teacher asked students to share what they knew about fractions and decimals. It was interesting to note that when students were asked to elaborate on their ideas, they often asked if they could draw or refer to the representations that were on the board. Student responses included these:

> Fractions can be equivalent, like 1/2 equals 3/6. [Drawing a picture on the board]
>
> Fractions and decimals can be the same thing. 1/2 = 0.5 [Writing the symbols on the board]
>
> You can say a decimal by reading it as a fraction like read 0.453 as 453/1000 [Connecting two representations]
>
> The left of the decimal is a whole, and the right of the decimal is the part of a whole. [Pointing to the place-value chart]

By allowing students to elaborate through drawing, writing on the board, or pointing to a visual element such as the place-value chart, they used different representations to demonstrate their understanding. At this point in the lesson, it was appropriate to review some new vocabulary with the students, especially the ELL students. Vocabulary included *compatible numbers, rounding, estimating, addends,* and *sum*.

Next, the teacher distributed 10 × 10 grids on cards called *decimal grids*. Each card had a different decimal value that had been shaded. Students worked in pairs to name the value using a fractional representation as well as a decimal representation. Students used sticky notes to write their representations and then brought them to the board. For example, one student said, "sixty hundredths," then wrote 0.60 = 60/100 on the chalkboard. Students demonstrated flexibility in their representations of these numbers; they could identify a pictorial representation and then use fractions and decimal numbers to name these pictures.

As a way of getting students ready for the main activity, the teacher asked students to find two cards that had a sum close to one whole. (See **fig. 3**.) One student brought up the cards 20/100 and 0.8. His classmates stated that these two cards summed to 100/100, which equaled 1. Another student selected cards 0.45 and 0.35. With the sum of 0.80, that student was asked to tell how far the sum was from a whole. He responded, “20/100, or 0.20.” This initial activity helped students use compatible numbers to add the two decimal numbers quickly and mentally and even subtract from a whole.

Next, the rules of the game were introduced by students. One student read the rules from a poster, and all students read the place values on the chart (ones, tenths, hundredths) as a way of including all students. The instructor demonstrated how the game worked by using any four of the 1, 3, 4, 5, and 8 cards. One student suggested using the 1, 3, 4, and 5 cards to create the following addition problem: 0.43 + 0.51 = 0.94. This student said that the tenths place was important when selecting the numbers. Another student suggested using 1, 4, 5, and 8 to create the following addition problem: 0.15 + 0.84 = 0.99. Another student responded that this sum was 1/100 away from 1 whole. (See **fig. 4**.)

Once students were familiar with the game, they could play with a partner at their own desks. Students used the task sheet to record the two addends as decimal numbers on a place-value chart (step 1). Next, they shaded in the sum on one or two of the 10 × 10 grids (step 2). Some students shaded in their sum as a complete picture. For example, if they had achieved 0.99, they shaded 99 little squares on one of the grids. Other students put each addend on a different 10 × 10 grid. For example, the left grid might contain the first addend (0.15 shaded) and the right grid might contain the second addend (0.84 shaded). Next, students were asked to write whether the sum was greater than or less than 1 (step 3). And finally, students had to determine how far away from 1 they were (step 4).

As students worked in pairs to play the game, the teachers observed their interactions with each other as well as with the task sheets. Before the class was over, the teacher brought the group back together. She asked students what strategies they used to get close to 1, then asked them to think about which value was more important in forming a number closest to 1 whole—the tenths place or the hundredths place.

Fig. 3 Finding compatible numbers with decimal grids, vocabulary cards, and other representations for decimal number sense

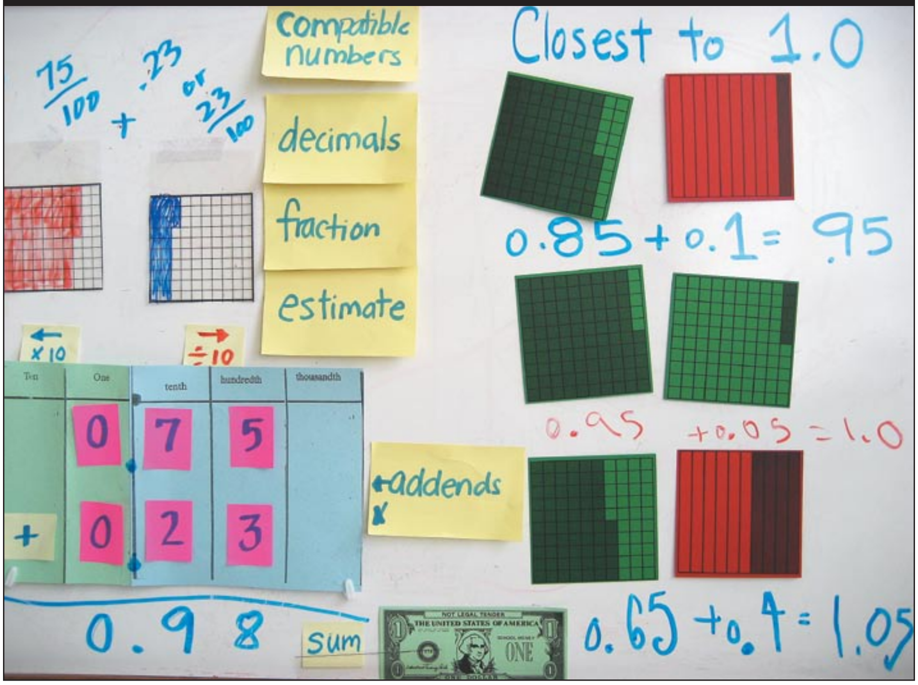
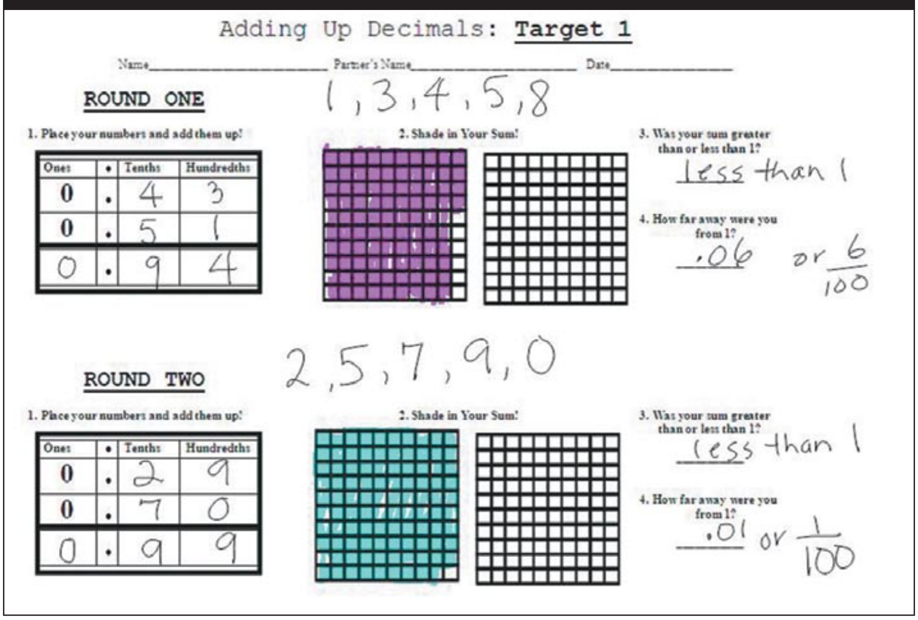


Fig. 4 Task sheet for recording thinking using multiple representations to build number sense



PHASE 3: REFLECTING ON DEVELOPMENT OF REPRESENTATIONAL FLUENCY

The debriefing process that occurred immediately after the lesson was an invaluable opportunity for teachers to engage in collaborative reflection. Some common themes that emerged in the conversation regarding representations are summarized here.

Representations encouraged flexibility of thinking and making generalizations. The use of the place-value chart and the base-ten representation helped students extend their understanding of our number system from their prior knowledge of whole numbers to decimal numbers. They were able to appreciate the relationship between the base-ten blocks (the unit, long, and flat) to the base-ten grids and the whole-number place-value chart to the decimal place-value chart. Teachers had been concerned that students would confuse the flat, which typically represents 100 when working with whole numbers, with the decimal grid, in which the flat now represented 1 whole. It was important for students to use the representations flexibly so that with their previous knowledge of base-ten relationships, they could understand that decimals are simply an extension of the base-ten number system and not a different system. In addition, the representations allowed students to make generalizations about the relationship of numbers and their place value by stating that the value of the digits was determined by the place. In playing the game, it was strategic for students to choose the digit for the tenths place more carefully than that for the hundredths place so that the sum would be closest to 1 whole.

Representations gave access to English language learners and special needs students. It was important to build mathematical vocabulary actively through meaningful activities that involved representing the word meaning linguistically and nonlinguistically with pictures, objects, and symbols. Using other forms of representations to aid the understanding of specific mathematics vocabulary helped ELL and special needs learners gain access to this lesson. For example, the pictorial 10×10 grids helped students calculate the decimal sums, state the decimals in words, and relate them to fractions. In addition, having monetary amounts like dollars, dimes, and pennies handy as an alternative model helped these students access their prior knowledge of decimal numbers as they connected their knowledge of informal mathematics in the formal classroom setting. In addition, having vocabulary words easily accessible on the board allowed these students to use academic terms, such as *addends, sum, tenths, hundredths,* and *difference,* as they talked with their classmates and teacher. Words are important representations used to communicate mathematical ideas that appear in word problems, problem solving, and in basic directions. Students categorized as ELL, who are learning mathematics, can benefit from teachers actively building academic vocabulary.

Representations were used to assess students' mathematics understanding. Since representations act as tools for communicating mathematical ideas, it is important that they are used to assess students' mathematical understanding. By using the hands-on materials and the task sheet, teachers were able to assess students' mathematical understanding through their representations. For example, observing how students arranged the five digits in making an addition statement closest to 1 was more revealing than correcting a worksheet with twenty problems. The teacher engaged students in conversations while they played, asking such questions as "Why did you decide to place the digit in the tenths place? What other digits might you consider? How would switching the digits affect the sum?" These probing questions caused students to think, make conjectures, and test a hypothesis and allowed the teacher to make anecdotal records of students' responses, which revealed their thinking and reasoning. The task sheet allowed students to connect pictorial representations and numeric representations explicitly, allowing them to form mental images of decimals.

Representations served as tools for communicating mathematical ideas. Representations can support connections, reasoning, communication, and problem solving. However, without promoting these mathematical ideas and verbalizing them in class discussion, the rich potential of learning can be lost. Classroom discourse and teacher questions helped students clarify their thinking and bridge representations with important mathematical learning. As students engaged in mathematical discourse using the various representations, the learning environment created an opportunity for students to internalize the concepts through concrete learning.

The multiple representations of decimal grids, the place-value chart, movable digits, and vocabulary and numeric symbols facilitated students' ability to communicate and make sense of mathematical ideas when working with a partner. The game and activities allowed students to work flexibly with these multiple representations and to translate among the different modes, thereby promoting the development of decimal number sense.

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